

# Higher geometric prequantum theory

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## Abstract

We develop the refinement of geometric prequantum theory to higher geometry (higher stacks), where a prequantization is given by a higher principal connection (a higher gerbe with connection). We show fairly generally how there is canonically a tower of higher gauge groupoids and Courant groupoids assigned to a higher prequantization, and establish the corresponding Atiyah sequence as an integrated Kostant-Souriau  $\infty$ -group extensions of higher Hamiltonian symplectomorphisms by higher quantomorphisms. We also exhibit the  $\infty$ -group cocycle which classifies this extension and discuss how its restrictions along Hamiltonian  $\infty$ -actions yield higher Heisenberg cocycles. In the special case of higher differential geometry over smooth manifolds we find the  $L_\infty$ -algebra extension of Hamiltonian vector fields which is the higher Poisson bracket of local observables and show that it is equivalent to the construction proposed by the second author in  $n$ -plectic geometry. Finally we indicate a list of examples of applications of higher prequantum theory in the extended geometric quantization of local quantum field theories and specifically in string geometry.

## Contents

<b>1</b>	<b>Motivation</b>	<b>2</b>
1.1	Traditional prequantum theory via slicing . . . . .	2
1.2	The need for higher prequantum theory . . . . .	5
<b>2</b>	<b>Survey</b>	<b>6</b>
2.1	Higher geometry . . . . .	6
2.1.1	Higher toposes of geometric $\infty$ -groupoids . . . . .	7
2.1.2	Higher geometric fiber bundles . . . . .	14
2.2	Higher gauge groupoids . . . . .	18
2.2.1	Bisections of higher groupoids . . . . .	18
2.2.2	Higher Atiyah groupoids . . . . .	19
2.3	Higher differential geometry . . . . .	23
2.3.1	Differential coefficients . . . . .	25
2.3.2	Differential moduli . . . . .	30
2.4	Higher differential gauge groupoids . . . . .	33
2.4.1	Higher quantomorphism- and Heisenberg-groupoids . . . . .	33
2.4.2	Higher Courant groupoids . . . . .	35
2.5	Higher integrated Kostant-Souriau extensions . . . . .	37
2.5.1	The quantomorphism $\infty$ -group extensions . . . . .	37
2.5.2	The higher Poisson bracket $L_\infty$ -extensions . . . . .	38
2.6	Examples . . . . .	42
2.6.1	Higher prequantum 2d WZW model and the smooth string 2-group . . . . .	42
2.6.2	Higher prequantum $nd$ Chern-Simons-type theories and $L_\infty$ -algebra cohomology . . . . .	45
2.6.3	Higher prequantum 2d Poisson Chern-Simons theory and quantum mechanics . . . . .	48
2.6.4	Higher prequantum 6d WZW-type models and the smooth fivebrane-6-group . . . . .	52

# 1 Motivation

## 1.1 Traditional prequantum theory via slicing

Traditional prequantum geometry is the differential geometry of smooth manifolds which are equipped with a twist in the form of a  $U(1)$ -principal bundle with a  $U(1)$ -principal connection. (See section II of [1] for a modern account.) In the context of geometric quantization [3] of symplectic manifolds these arise as *prequantizations* (whence the name): lifts of the symplectic form from de Rham cocycles to differential cohomology. Equivalently, prequantum geometry is the *contact geometry* of the total spaces of these bundles, equipped with their Ehresmann connection 1-form [4]. Prequantum geometry studies the automorphisms of prequantum bundles covering diffeomorphisms of the base – the prequantum operators or contactomorphisms – and the action of these on the space of sections of the associated line bundle – the prequantum states. This is an intermediate step in the genuine geometric quantization of symplectic manifolds, which is obtained by “dividing the above data in half” by a choice of polarization. While polarizations do play central role in geometric quantum theory, for instance in the orbit method in geometric representation theory [5], to name just one example, geometric prequantum theory is of interest in its own right. For instance the quantomorphism group naturally provides a non-simply connected Lie integration of the Poisson bracket Lie algebra of the underlying symplectic manifold and the pullback of this extension along Hamiltonian actions induces central extensions of infinite-dimensional Lie groups (see for instance [6, 7]). Moreover, the quantomorphism group comes equipped with a canonical injection into the group of bisections of the groupoid which integrates the Atiyah Lie algebroid associated with the given principal bundle (this we discuss below in 2.4). These are fundamental objects in the study of principal bundles over manifolds.

We observe now that all this has a simple natural reformulation in terms of the maps into the smooth moduli stacks [8, 11] that classify – better: modulate – principal bundles and principal connections. This reformulation exhibits an abstract characterization of prequantum geometry which immediately generalizes to higher geometric contexts richer than traditional differential geometry [D]. In 1.2 below we say why such a generalization is indeed desirable and in 2 we survey constructions and results in higher geometric prequantum theory.

To start with, if we write  $\Omega_{\text{cl}}^2$  for the sheaf of smooth closed differential 2-forms (on the site of all smooth manifolds), then by the Yoneda lemma a closed (for instance symplectic) 2-form  $\omega$  on a smooth manifold  $X$  is equivalently a map of sheaves  $\omega : X \longrightarrow \Omega_{\text{cl}}^2$ . It is useful to think of this as a simple first instance of moduli stacks:  $\Omega_{\text{cl}}^2$  is the universal moduli stack of smooth closed 2-forms.

Similarly but more interestingly, there is a smooth moduli stack of circle-principal connections [8] (a stack of groupoids on the site of smooth manifolds). This we denote by  $\mathbf{BU}(1)_{\text{conn}}$  in order to indicate that it is a differential refinement of the universal moduli stack  $\mathbf{BU}(1)$  of just  $U(1)$ -principal connections, which in turn is a smooth refinement of the traditional classifying space  $BU(1) \simeq K(\mathbb{Z}, 2)$  of just equivalence classes of such bundles. Hence  $\mathbf{BU}(1)_{\text{conn}}$  is the “smooth homotopy 1-type” which is uniquely characterized by the fact that maps  $X \rightarrow \mathbf{BU}(1)_{\text{conn}}$  from a smooth manifold  $X$  are equivalently circle-principal connections on  $X$ , and that homotopies between such maps are equivalently smooth gauge transformations between such connections. This is a refinement of  $\Omega_{\text{cl}}^2$ : the map which sends a circle-principal connection to its curvature 2-form constitutes a map of universal moduli stacks  $F_{(-)} : \mathbf{BU}(1)_{\text{conn}} \longrightarrow \Omega_{\text{cl}}^2$ , hence a universal invariant 2-form on  $\mathbf{BU}(1)_{\text{conn}}$ . This universal curvature form characterizes traditional prequantization: for  $\omega \in \Omega_{\text{cl}}^2(X)$  a (pre-)symplectic form as above, a *prequantization* of  $(X, \omega)$  is equivalently a lift  $\nabla$  in the diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad \nabla \quad} & \mathbf{BU}(1)_{\text{conn}} \\ \omega \searrow & & \swarrow F_{(-)} \\ & \Omega_{\text{cl}}^2 & \end{array},$$

where the commutativity of the diagram expresses the traditional prequantization condition  $\omega = F_{\nabla}$ .

A triangular diagram as above may naturally be interpreted as exhibiting a map *from*  $\omega$  *to*  $F_{(-)}$  *in the slice topos over*  $\Omega_{\text{cl}}^2$ . This means that the map  $F_{(-)}$  is itself a universal moduli stack – the universal moduli stack of prequantizations. As such,  $F_{(-)}$  lives not in the topos over all smooth manifolds, but in its slice over  $\Omega_{\text{cl}}^2$ , which is the topos of smooth stacks equipped with a map into  $\Omega_{\text{cl}}^2$ .

Now given a prequantization  $\nabla$ , then a *quantomorphism* or *integrated prequantum operator* is traditionally defined to be a pair  $(\phi, \eta)$ , consisting of a diffeomorphism  $\phi : X \xrightarrow{\sim} X$  together with an equivalence of prequantum connections  $\eta : \phi^* \nabla \xrightarrow{\sim} \nabla$ . A moment of reflection shows that such a pair is equivalently again a triangular diagram, now as on the right of

$$\mathbf{QuantMorph}(\nabla) = \left\{ \begin{array}{l} \phi \in \text{Diff}(X), \\ \eta : \phi^* \nabla \xrightarrow{\sim} \nabla \end{array} \right\} \simeq \left\{ \begin{array}{ccc} X & \xrightarrow{\quad \phi \quad} & X \\ & \swarrow \eta & \searrow \\ & \nabla & \nabla \\ & \mathbf{BU}(1)_{\text{conn}} & \end{array} \right\}.$$

This also makes the group structure on these pairs manifest – the *quantomorphism group*: it is given by the evident pasting of triangular diagrams. In this form, the quantomorphism group is realized as an example of a very general construction that directly makes sense also in higher geometry: it is the automorphism group of a modulating morphism regarded as an object in the slice topos over the corresponding moduli stack – a relative automorphism group. Also in this form the central property of the quantomorphism group – the fact that over a connected manifold it is a  $U(1)$ -extension of the group of Hamiltonian symplectomorphisms – is revealed to be just a special case of a very general extension phenomenon, expressed by the schematic diagrams below:

$$\begin{array}{ccccc} U(1) & \rightarrow & \mathbf{QuantMorph}(\nabla) & \rightarrow & \mathbf{HamSymp}(\nabla) \\ \left\{ \begin{array}{c} X \\ \nabla \left( \begin{array}{c} \Leftarrow \\ \Leftarrow \end{array} \right) \nabla \\ \mathbf{BU}(1)_{\text{conn}} \end{array} \right\} & \rightarrow & \left\{ \begin{array}{ccc} X & \xrightarrow{\quad \sim \quad} & X \\ & \swarrow & \searrow \\ & \nabla & \nabla \\ & \mathbf{BU}(1)_{\text{conn}} & \end{array} \right\} & \rightarrow & \left\{ X \xrightarrow{\quad \sim \quad} X \right\}. \end{array}$$

Our main theorems in 2.5 below are a general account of canonical extensions induced by (higher) automorphism groups in slices over (higher, differential) moduli stacks in this fashion.

This  $U(1)$ -extension is the hallmark of quantization: under Lie differentiation the above sequence of (infinite-dimensional) Lie groups turns into the extension of Lie algebras

$$i\mathbb{R} \rightarrow \mathfrak{Poisson}(X, \omega) \rightarrow \mathfrak{X}_{\text{Ham}}(X, \omega)$$

that exhibits the Poisson bracket Lie algebra of the symplectic manifold as an  $i\mathbb{R} \simeq \text{Lie}(U(1))$ -extension of the Lie algebra of Hamiltonian vector fields on  $X$  – the *Kostant-Souriau extension* (e.g [1, section 2.3]). If we write  $\hbar \in \mathbb{R}$  for the canonical basis element (“Planck’s constant”) then this expresses the quantum deformation of “classical commutators” in  $\mathfrak{X}_{\text{Ham}}(X, \omega)$  by the central term  $i\hbar$ .

More widely known than the quantomorphism groups of all prequantum operators are a class of small subgroups of them, the *Heisenberg groups* of translational prequantum operators: if  $(X, \omega)$  is a symplectic vector space of dimension  $2n$ , regarded as a symplectic manifold, then the translation group  $\mathbb{R}^{2n}$  canonically acts on it by Hamiltonian symplectomorphisms, hence by a group homomorphism  $\mathbb{R}^{2n} \rightarrow \mathbf{HamSymp}(\nabla)$ . The pullback of the above quantomorphism group extension along this map yields a  $U(1)$ -extension of  $\mathbb{R}^{2n}$ , and this is the traditional Heisenberg group  $H(n, \mathbb{R})$ . More generally, for  $(X, \omega)$  any (prequantized) symplectic manifold and  $G$  any Lie group, one considers *Hamiltonian  $G$ -actions*: smooth group homomorphisms  $\phi : G \rightarrow \mathbf{HamSymp}(\nabla)$ . Pulling back the quantomorphism group extension now yields a  $U(1)$ -extension of  $G$  and this we may call, more generally, the Heisenberg group extension induced by the Hamiltonian  $G$ -action:

$$U(1) \rightarrow \mathbf{Heis}_{\phi}(\nabla) \rightarrow G.$$

The crucial property of the quantomorphism group and any of its Heisenberg subgroups, at least for the purposes of geometric quantization, is that these are canonically equipped with an action on the space of prequantum states (the space of sections of the complex line bundle which is associated to the prequantum bundle), this is the action of the *exponentiated prequantum operators*. Under an *integrated moment map*, – a group homomorphism  $G \rightarrow \mathbf{QuantMorph}(\nabla)$  covering a Hamiltonian  $G$ -action – this induces a representation of  $G$  on the space of prequantum states. After a choice of polarization this is the construction that makes geometric quantization a valuable tool in geometric representation theory.

This action of prequantum operators on prequantum states is naturally interpreted in terms of slicing, too: A prequantum operator is traditionally defined to be a function  $H \in C^\infty(X)$  with action on prequantum states  $\psi$  traditionally given by the formula

$$O_H : \psi \mapsto i\nabla_{v_H}\psi + H \cdot \psi,$$

where the first term is the covariant derivative of the prequantum connection along the Hamiltonian vector field corresponding to  $H$ . To see how this formula together with its Lie integration, falls out naturally from the perspective of the slice over the moduli stack, write  $\mathbb{C} // U(1)$  for the quotient stack of the canonical 1-dimensional complex representation of the circle group, and observe that this comes equipped with a canonical map  $\rho : \mathbb{C} // U(1) \longrightarrow * // U(1) \simeq \mathbf{BU}(1)$  to the moduli stack of circle-principal bundles. This is the universal complex line bundle over the moduli stack of  $U(1)$ -principal bundles, and it has a differential refinement compatible with that of its base stack to a map  $\rho_{\text{conn}} : \mathbb{C} // U(1)_{\text{conn}} \longrightarrow \mathbf{BU}(1)_{\text{conn}}$ . Now one can work out that maps  $\psi : \nabla \rightarrow \rho_{\text{conn}}$  in the slice over  $\mathbf{BU}(1)_{\text{conn}}$  are equivalently sections of the complex line bundle  $P \times_{U(1)} \mathbb{C}$  which is  $\rho$ -associated to the  $U(1)$ -principal prequantum bundle:

$$\Gamma_X (P \times_{U(1)} \mathbb{C}) \simeq \left\{ \begin{array}{ccc} X & \xrightarrow{\psi} & \mathbb{C} // U(1) \\ & \swarrow \nabla & \searrow \rho_{\text{conn}} \\ & \mathbf{BU}(1)_{\text{conn}} & \end{array} \right\}.$$

With this identification, the action of quantomorphisms on prequantum states

$$(O_h, \psi) \mapsto O_h(\psi)$$

is simply the precomposition action in the slice  $\mathbf{H}_{/\mathbf{BU}(1)}$ , hence the action by pasting of triangular diagrams in  $\mathbf{H}$ :

$$\left( \begin{array}{ccc} X & \xrightarrow{\phi} & X \\ & \swarrow \nabla & \searrow \nabla \\ & \mathbf{BU}(1)_{\text{conn}} & \end{array} \right), \quad \begin{array}{ccc} X & \xrightarrow{\psi} & \mathbb{C} // U(1)_{\text{conn}} \\ & \swarrow \nabla & \searrow \rho_{\text{conn}} \\ & \mathbf{BU}(1)_{\text{conn}} & \end{array} \quad \mapsto \quad \begin{array}{ccccc} X & \xrightarrow{\phi} & X & \xrightarrow{\psi} & \mathbb{C} // U(1)_{\text{conn}} \\ & \swarrow \nabla & \downarrow & \searrow \rho_{\text{conn}} & \\ & \mathbf{BU}(1)_{\text{conn}} & & & \end{array}$$

Once formulated this way as the geometry of stacks in the higher slice topos over the smooth moduli stack of principal connections, it is clear that there is a natural generalization of traditional prequantum geometry, hence of regular contact geometry, obtained by interpreting these diagrams in higher differential geometry with smooth moduli stacks of principal bundles and principal connections refined to higher smooth moduli stacks [8, 10, 11]. Moreover, by carefully abstracting the minimum number of axioms on the ambient toposes actually needed in order to express the relevant constructions (this we discuss in 2.3) one obtains generalizations to various other flavors of higher/derived geometry, such as higher/derived supergeometry.

Just as traditional prequantum geometry and contact geometry is of interest in itself, this natural refinement to higher geometry is of interest in itself, and is one motivation for studying higher prequantum geometry. For instance in 2.6 we indicate how various higher central extensions of interest in string geometry can be constructed as higher Heisenberg-group extensions in higher prequantum geometry.

But the strongest motivation for studying traditional prequantum geometry is, as the name indicates, as a means in quantum mechanics and quantum field theory. In the next section 1.2 we discuss how the generalization of traditional geometric quantization to local (“extended”) quantum field theory involves higher geometric prequantum theory.

## 1.2 The need for higher prequantum theory

Important examples of prequantum bundles turn out to be transgressions of higher geometric bundles to mapping spaces or more generally to mapping stacks. A classical example is the canonical prequantum bundle over the loop group  $LG$  of a compact simply connected group  $G$  – the prequantum bundle whose geometric quantization induces the positive energy representations of  $LG$  [12]. This prequantum bundle is the transgression of the canonical bundle gerbe (a  $BU(1)$ -principal 2-bundle) on  $G$  (e.g. [1, chapter VI]). Another example is the ordinary prequantum bundle of Chern-Simons theory on the space of (flat)  $G$ -principal connections over a surface, which is the transgression of a canonical  $B^2U(1)$ -principal 3-connection (see 2.3 below) over the universal moduli stack of  $G$ -principal connections [13, 14]. In fact, the previous example is a sub-phenomenon of this one, see the discussion in 2.6.1 below.

A central theme in higher geometry is that transgression (in general) loses information and that it is better to study the higher geometric pre-images before transgression. An archetypical example of this phenomenon that has motivated many developments that we are building on are *string structures* on a space and their transgression to some kind of spin structures on loop space. (A review and careful analysis of this case is in [15].) This alone suggests that when faced with a prequantum bundle that is the transgression of a higher bundle, one should look for some kind of prequantization of that higher bundle and hence *de-transgress* the traditional prequantum structure to higher cohomology/higher geometry. For the geometric quantization of loop groups by the orbit method [5, 12] this was suggested twenty years ago in [1, p. 249]: “The main open question seems to be to obtain the representation theory of  $\hat{L}G$  from the canonical sheaf of groupoids on  $G$ ... we ask whether some quantization method exists based on the sheaf of groupoids”. Here *sheaf of groupoids* means *stack of groupoids*, and although there has been progress, see our discussion in 2.6.1 below, the question remains open. The above question was also a motivation behind the refinement of multisymplectic geometry to homotopy theory developed in [16]. This led to a higher Bohr-Sommerfeld-like geometric quantization procedure for manifolds equipped with closed integral 3-forms [2, Chap. 7]. This procedure includes not just higher prequantization, but higher notions of real polarizations, (i.e. *2-polarizations*) as well as quantum 2-states. The output is a certain kind of linear category, which, for example, is the representation category of a Lie group. However, many aspects introduced in this previous work need further development, some of which is given by the results we present here.

The case for higher prequantization can be made even more forcefully by considering *quantized* field theories. The classification of local (“extended”) topological quantum field theories (TQFTs) in [17] shows that an  $n$ -dimensional local quantum field theory assigns not just a vector space of quantum states in codimension 1, but assigns some kind of *k-module of quantum k-states* in each codimension  $k$  (see [18] for a survey of the formalization of quantum field theory in higher category theory). And for topological quantum field theories all these assignments are entirely determined by the assignment of a single  $n$ -module of  $n$ -states over the point in codimension  $n$ . This localized extension of TQFTs is thought to reveal deep structures in quantum field theories of relevance in low dimensional topology, such as Chern-Simons theories and their various siblings and higher generalizations; a recent review is in [19].

What has been missing is a refinement of the process of geometric quantization that goes along with this local and homotopy-theoretic refinement of quantum field theory, and hence a means to construct and control such extended TQFTs from and by prequantum geometric data. But the TQFT classification theorem suggests that for geometrically quantizing an  $n$ -dimensional quantum field theory, one should have a *prequantum n-bundle* of the theory over the higher moduli stack of fields [14, 27], a notion of (polarized) sections of a suitable associated fiber  $\infty$ -bundle, and finally an identification of the collection of these with the *n-module of quantum n-states* that (thanks to [17]) defines the local TQFT. As in the traditional story, this process should proceed in two steps: first higher prequantization, then higher polarization. Here we are concerned with the first step:

differential geometric (Lagrangian) data	$\xrightarrow{\text{quantization}}$	quantum field theory
traditional geometric prequantum theory		quantum mechanics
higher geometric prequantum theory		local (extended) QFT

At the level of just (pre-)symplectic differential form data, the physics literature describes this need to pass to higher codimension as the problem of “non-covariance of canonical quantization”, which refers to the choice of spatial slices of spacetime involved in the construction of the space of states of an  $n$ -dimensional quantum field theory in codimension 1. There are two established techniques for dealing with this by a “covariant” procedure that refines (pre-)symplectic differential 2-forms on spaces of fields over a chosen Cauchy surface by differential  $(n + 1)$ -forms on the jet bundle of the field bundle: these are *multisymplectic geometry* and the *covariant phase space method*. A decent review of both of these and a discussion of how they are related is in [20].

The remaining open question used to be: what is to multisymplectic forms as geometric prequantum theory is to symplectic forms? For instance: what replaces the Poisson bracket Lie algebra as we pass from observables given by functions on phase space to *local* observables given by differential forms (currents) on the extended phase space? In references such as [21] it is observed that the collection of differential form observables in such a context inherits the structure of a (graded) Lie algebra only after exerting some force: after restricting to smaller subspaces and/or after quotienting out terms that would otherwise violate the Jacobi identity. The crucial insight of [16] was that these terms that violate the Jacobi identity are *coherent* and hence instead of being a nuisance are part of a natural but higher structure: Hamiltonian  $(n - 1)$ -form observables together with all lower degree form observables (not discarding or quotienting out any of them) constitute not a Poisson bracket Lie algebra but its homotopy-theoretic refinement: an  $L_\infty$ -algebra of *local observables* (recalled as def. 2.5.8 below). This is exactly what one expects to see in a higher geometric version of geometric quantization by the above reasoning.

One purpose of the present article is to show that these higher-Poisson bracket  $L_\infty$ -algebras of local observables are part of a general and robust higher geometric prequantum theory.

## 2 Survey

We start with briefly recalling some background in higher geometry/higher topos theory in 2.1 to set the scene. Then in 2.2 we observe that in this context the traditional notion of the Atiyah groupoid (“gauge groupoid”) of a principal bundle has a fundamental and general formulation. This is the archetype of all considerations to follow.

In 2.3 we review the axiomatic refinement of higher geometry to higher differential geometry/differential cohomology and state the *concretification* of higher moduli stacks of principal connections. This is the crucial technical ingredient in the definition and for the properties of the quantomorphism  $\infty$ -group, which we then introduce as part of a sequence of higher Atiyah-, higher Courant- and higher Heisenberg groupoids in 2.4. Finally we state the main theorem, the refinement of the Kostant-Souriau-extension to higher geometry, in 2.5.

Examples of higher prequantum geometries of some classes of quantum field theories in 2.6 serve as an outlook on the applications of higher geometric prequantum theory.

### 2.1 Higher geometry

In the introduction in 1.1 we indicated how traditional geometric prequantum theory has a natural formulation in terms of stacks of groupoids over the site of smooth manifolds. Accordingly, higher geometric prequantum theory has a natural formulation in terms of stacks of *higher groupoids* (homotopy types) on a site of geometric test spaces. Conversely, since a collection of higher stacks forms a context called an  $\infty$ -topos, and since these are particularly well-behaved contexts for formulating geometric theories, our formulation of higher prequantum geometry is guided by notions that are natural in higher topos theory. Such an axiomatic approach guarantees robust general notions: everything that we discuss here makes sense and holds in *every*



$\infty$ -topos whatsoever, be it one that models higher/derived differential geometry, complex geometry, analytic geometry, supergeometry etc.

The only constraining assumption that we need later on arises in 2.3 below, when we turn from plain geometric cohomology to *differential cohomology*. For that to make sense we need to impose a minimum of axioms that guarantees that the ambient  $\infty$ -topos supports not only a good notion of fiber/principal  $\infty$ -bundles, as every  $\infty$ -topos does, but also of connections on such bundles.

This section is a brief commented list of some basic constructions and facts in higher geometry/higher topos theory which we need below; the foundational aspects in 2.1.1 taken from [28], and the fiber bundle and representation theory in 2.1.2 taken from [10, 11].

### 2.1.1 Higher toposes of geometric $\infty$ -groupoids

The notion of  $\infty$ -topos [28] combines geometry with homotopy theory, hence with higher gauge symmetry: given a category  $C$  of geometric test spaces (hence equipped with a Grothendieck topology), the  $\infty$ -topos of  $\infty$ -stacks over it, denoted  $\mathrm{Sh}_\infty(C)$ , is the homotopy theory obtained by taking the category  $[C^{\mathrm{op}}, \mathrm{KanCplx}]$  of Kan-complex valued presheaves on  $C$  and then universally turning *local* homotopy equivalence between such presheaves (local as seen by the Grothendieck topology) into global homotopy equivalences. This process is called *simplicial localization* (see [11] for a review and further details), denoted by the right hand side of

$$\mathrm{Sh}_\infty(C) \simeq L_{\mathrm{he}}[C^{\mathrm{op}}, \mathrm{KanCplx}].$$

More generally, for  $\mathcal{C}$  any category and  $W \subset \mathrm{Mor}(\mathcal{C})$  a collection of morphisms, there is the homotopy theory of the simplicial localization  $L_W \mathcal{C}$  obtained by universally turning the morphisms in  $W$  into homotopy equivalences. This is the  $\infty$ -category induced by  $(\mathcal{C}, W)$ . A homotopy-theoretic functor  $L_{W_1} \mathcal{C}_1 \rightarrow L_{W_2} \mathcal{C}_2$  between such homotopy-theoretic categories is an  $\infty$ -functor. If this is induced from an ordinary functor  $\mathcal{C}_1 \rightarrow \mathcal{C}_2$  it is also called a (total) *derived functor*. The example of this that we use prominently is the Dold-Kan functor below in remark 2.1.4.

**Example 2.1.1.** The basic example is the  $\infty$ -topos  $\infty\mathrm{Grpd}$  of  $\infty$ -groupoids (hence of geometrically discrete  $\infty$ -groupoids!). This is presented equivalently by the simplicial localization of the category  $\mathrm{KanCplx}$  of Kan complexes at the homotopy equivalences, or of the category  $\mathrm{Top}$  of (compactly generated weakly Hausdorff) topological spaces at the weak homotopy equivalences:

$$\infty\mathrm{Grpd} \simeq L_{\mathrm{he}}\mathrm{KanCplx} \simeq L_{\mathrm{we}}\mathrm{Top}.$$

Hence this is just traditional homotopy theory thought of as the  $\infty$ -topos of geometrically discrete  $\infty$ -groupoids.

**Example 2.1.2.** The most immediate choice of  $\infty$ -topos which subsumes traditional differential geometry, foliation/orbifold theory and Lie groupoid/differentiable stack theory is that of  $\infty$ -stacks over the site of smooth manifolds with its standard Grothendieck topology of open covers. We write

$$\mathrm{Smooth}\infty\mathrm{Grpd} \simeq \mathrm{Sh}_\infty(\mathrm{SmthMfd}) \simeq L_{\mathrm{he}}[\mathrm{SmoothMfd}^{\mathrm{op}}, \mathrm{KanCplx}]$$

for this  $\infty$ -topos. The ordinary category of smooth manifolds is faithfully embedded into this  $\infty$ -topos, as is the collection of Lie groupoids with generalized/Morita-morphisms between them (“differentiable stacks”).

More in detail, a Lie groupoid  $\mathcal{G} = \left( \mathcal{G}_1 \begin{smallmatrix} \xrightarrow{t} \\ \xleftarrow{s} \end{smallmatrix} \mathcal{G}_0 \right)$  is identified, up to equivalence, with the presheaf of Kan complexes given by

$$\underline{\mathcal{G}} : U \mapsto N \left( C^\infty(U, \mathcal{G}_1) \begin{smallmatrix} \xrightarrow{C^\infty(U, t)} \\ \xleftarrow{C^\infty(U, s)} \end{smallmatrix} C^\infty(U, \mathcal{G}_0) \right)$$

for every smooth manifold  $U$ , where  $N : \mathbf{Grpd} \rightarrow \mathbf{KanCplx}$  is the nerve functor. See [30] for how orbifolds and foliations are special cases of Lie groupoids and hence are similarly embedded into  $\mathbf{Smooth}\infty\mathbf{Grpd}$ . Basic tools for explicit computations with objects in  $\mathbf{Smooth}\infty\mathbf{Grpd}$  and similar contexts of higher geometry are discussed in [8, 11, D].

**Remark 2.1.3.** As in traditional homotopy theory, when we draw a commuting diagram of morphisms in an  $\infty$ -category, it is always understood that they commute up to a specified homotopy. We will often notationally suppress these homotopies that fill diagrams, except if we want to give them explicit labels. For instance, in the figure below, the diagram of morphisms in an  $\infty$ -category on the left hand side always means the more explicit diagram displayed on the right hand side:

$$\begin{array}{ccc} X & \xrightarrow{\quad} & Y \\ & \searrow & \swarrow \\ & A & \end{array} \quad := \quad \begin{array}{ccc} X & \xrightarrow{\quad} & Y \\ & \searrow \quad \swarrow & \\ & A & \end{array} .$$

In the same spirit all the universal constructions that we mention in the following refer to their homotopy-correct version. Notably fiber products in the following always are homotopy fiber products. With homotopies thus understood, most of the familiar basic facts of category theory generalize verbatim to  $\infty$ -category theory. For instance a basic fact that we make repeated use of is the pasting law for homotopy pullbacks: if we have two adjacent square diagrams and the right square is a homotopy pullback, then the left square is also such a pullback if and only if the total rectangle is.

**Remark 2.1.4** (generalized nonabelian sheaf cohomology). For  $X, A \in \mathbf{H}$  any two objects in an  $\infty$ -topos, we have an  $\infty$ -groupoid  $\mathbf{H}(X, A) \in \infty\mathbf{Grpd}$  consisting of morphisms from  $X$  to  $A$ , homotopies between such morphisms and higher homotopies between these, etc. We may think of this as the  $\infty$ -groupoid of cocycles, coboundaries and higher coboundaries on  $X$  with coefficients in  $A$ . The set of connected components of this  $\infty$ -groupoid

$$H(X, A) := \pi_0 \mathbf{H}(X, A)$$

is the *cohomology* of  $X$  with coefficients in  $A$ . This notion of cohomology in an  $\infty$ -topos unifies abelian sheaf cohomology with the generalized cohomology theories of algebraic topology and generalizes both to nonabelian cohomology that classifies higher principal bundles in  $\mathbf{H}$  (this we come to below in 2.1.2). Hence the *homotopy category*  $H$  of an  $\infty$ -topos  $\mathbf{H}$  may be thought of as a *generalized nonabelian sheaf cohomology theory*: the fact that it is a sheaf cohomology theory means that it encodes “geometric cohomology”, for instance “smooth cohomology” in example 2.1.2. Ordinary abelian sheaf cohomology is reproduced as the special case where the coefficient object  $A \in L_{\text{he}}[C^{\text{op}}, \mathbf{KanCplx}]$  is in the essential image of the Dold-Kan correspondence

$$\text{DK} : \text{Ch}_{\bullet \geq 0}(\text{Ab}) \xrightarrow{\simeq} \text{Ab}^{(\Delta^{\text{op}})} \xrightarrow{\text{forget}} \mathbf{KanCplx} ,$$

which regards a sheaf of chain complexes of abelian groups equivalently as a sheaf of simplicial abelian groups (whose normalized chain complex is the original complex), hence in particular as a sheaf of Kan complexes.

A crucial point of  $\infty$ -toposes is that they share the general abstract properties of classical homotopy theory in  $\infty\mathbf{Grpd} \simeq L_{\text{whe}}\mathbf{Top}$  (example 2.1.1). In our discussion of higher prequantum geometry starting in 2.2 we need specifically the following three technical aspects of homotopy theory in  $\infty$ -toposes:

1. Moore-Postnikov-Whitehead-theory;
2. relative theory over a base and base change;
3. looping and delooping.

In the remainder of this section we state the corresponding definitions and results that are used later on. The reader not interested in this level of technical detail should maybe skip ahead and come back here as need be.



There is a notion of *homotopy groups*  $\pi_n$  of objects in  $\mathbf{H}$ , however these are not groups in  $\mathbf{Set}$  but group objects in the 1-topos (sheaf topos) of 0-truncated objects of  $\mathbf{H}$ . With respect to these homotopy sheaves there is Moore-Postnikov-Whitehead theory:

**Remark 2.1.5.** An object  $A \in \mathbf{H}$  is called *n-truncated* if for all  $X \in \mathbf{H}$  the  $\infty$ -groupoid  $\mathbf{H}(X, A)$  is a homotopy  $n$ -type. The  $n$ -truncated objects in  $\mathbf{H} = L_{\text{lwe}}[C^{\text{op}}, \text{KanCplx}]$  are the *stacks of n-groupoids* on  $C$ . For  $n = 1$  these are ordinary stacks and for  $n = 0$  these are ordinary sheaves on  $C$ .

**Proposition 2.1.6.** *The full sub- $\infty$ -category of  $n$ -truncated objects  $\tau_{\leq n}\mathbf{H} \hookrightarrow \mathbf{H}$  in an  $\infty$ -topos is reflectively embedded, which means that there is an idempotent truncation projection  $\tau_n : \mathbf{H} \rightarrow \mathbf{H}$  which sends an arbitrary  $\infty$ -stack  $X$  to its universal approximation by an  $n$ -truncated object  $\tau_n X$ , the  $n$ th Postnikov stage of  $X$  as seen in  $\mathbf{H}$ .*

More generally given a morphism  $f : X \rightarrow Y$  in  $\mathbf{H}$ , there is a tower of factorizations

$$\begin{array}{ccc}
 & \vdots & \\
 & \text{im}_3(f) & \\
 & \downarrow & \\
 & \text{im}_2(f) & \\
 & \downarrow & \\
 X & \xrightarrow{\quad \text{im}_1(f) \hookrightarrow \quad} & Y \\
 & \searrow f & \\
 & & 
 \end{array}$$

with the property that for all  $n \in \mathbb{N}$  the morphism  $X \longrightarrow \text{im}_n(f)$  is an epimorphism on  $\pi_0$ , an isomorphism on  $\pi_{<n-1}$ , and that  $\text{im}_n(f) \longrightarrow Y$  is an injection on  $\pi_{n-1}$  and an isomorphism on all  $\pi_{\geq n}$ .

This is part of [28, section 5.5.6 and 6.5].

**Definition 2.1.7.** We call the objects  $\text{im}_n(f)$  in prop. 2.1.6 the *n-image* of  $f$  and say that morphisms of the form  $X \longrightarrow \text{im}_n(f)$  are *n-epimorphisms* and that morphisms of the form  $\text{im}_n(f) \longrightarrow Y$  are *n-monomorphisms* (in [28] these are called  $(n-1)$ -connective and  $(n-2)$ -truncated morphisms, respectively).

For  $n = 1$  the  $n$ -image factorization has a useful more explicit characterization:

**Proposition 2.1.8.** *For  $f : X \rightarrow Y$  a morphism in an  $\infty$ -topos  $\mathbf{H}$ , consider the homotopy-colimiting cocone under its Čech nerve simplicial diagram as indicated in the top row of the following diagram*

$$\begin{array}{ccc}
 \cdots \cdots \cdots X \times_Y X \times_Y X \rightrightarrows X \times_Y X \rightrightarrows X & \xrightarrow{p} & \left( \lim_{\rightarrow n} X^{\times_Y^{n+1}} \right) \simeq \text{im}_1(f) \\
 & \searrow f & \downarrow i \\
 & & Y
 \end{array}$$

Since  $f : X \rightarrow Y$  canonically extends to a homotopy cocone under its own Čech nerve, the universal property of the  $\infty$ -colimit induces a vertical dashed map  $i$ , as indicated. The resulting factorization of  $f$  is its 1-image factorization, as shown.

Another important aspect of  $\infty$ -toposes which is familiar both from traditional geometry as well as from traditional homotopy theory is the possibility of working relatively over a base object, the construction which

we amplified above in 1.1: given an object  $X \in \mathbf{H}$ , an *object over*  $X$  is just a map  $E \rightarrow X$  into  $X$ , and the collection of all of these with maps between them that fix  $X$  is written  $\mathbf{H}_{/X}$  and called the *slice  $\infty$ -topos* over  $X$ .

**Proposition 2.1.9.** *For all  $X \in \mathbf{X}$  the slice  $\mathbf{H}_{/X}$  is again an  $\infty$ -topos and the  $\infty$ -functor  $\sum_X : (E \rightarrow X) \mapsto E$  is the left part of an adjoint triple of base change  $\infty$ -functors:*

$$\left( \sum_X \dashv X \times (-) \dashv \prod_X \right) : \mathbf{H}_{/X} \begin{array}{c} \xrightarrow{\sum_X} \\ \xleftarrow{X \times (-)} \\ \xrightarrow{\prod_X} \end{array} \mathbf{H} .$$

This is [28, prop. 6.3.5.1].

**Remark 2.1.10.** Using the right adjointness of  $\prod_X$  in prop. 2.1.9 one finds that it sends a bundle  $E \rightarrow X$  to its space of sections, regarded naturally as a geometric  $\infty$ -groupoid itself, hence as an object of  $\mathbf{H}$ :

$$\Gamma_X(E) \simeq \prod_X E \in \mathbf{H} .$$

The underlying discrete  $\infty$ -groupoid of sections (forgetting the geometric structure) is as usual given by further evaluation on the point

$$\Gamma_X(E) \simeq \Gamma(\Gamma_X(E)) \in \infty\text{Grpd} .$$

This is the  $\infty$ -groupoid whose objects are naturally identified with sections  $\sigma$  in

$$\begin{array}{ccc} & & E \\ & \nearrow \sigma & \downarrow \\ X & \xlongequal{\quad} & X \end{array} ,$$

whose morphisms are homotopies of such sections, etc. The geometric  $\infty$ -groupoid of sections  $\Gamma_X(E) \in \mathbf{H}$  plays a central role in the discussion of genuine higher prequantum geometry starting below in 2.2.

**Remark 2.1.11.** Given  $A \in \mathbf{H}$ , the intrinsic cohomology, as in remark 2.1.4, of the slice  $\infty$ -topos  $\mathbf{H}_{/A}$  as in prop. 2.1.9 is equivalently *twisted cohomology* in  $\mathbf{H}$  with twist coefficients  $A$ : a domain object  $\mathcal{X} = (X \xrightarrow{\phi} A) \in \mathbf{H}_{/A}$  is equivalently an object  $X \in \mathbf{H}$  equipped with a twisting cocycle  $\phi$ , and a codomain object  $\mathcal{A} = (E \rightarrow A)$  is equivalently a *local coefficient bundle*. See [10, section 4.3] for a general abstract account of this and [14, 26] for examples relevant to higher prequantum theory. This observation, combined with our discussion in 2.4, implies that higher prequantum geometry is, equivalently, the geometry of spaces equipped with a (differential) cohomological twist. An archetypical example of this general phenomenon is the identification of the (pre-)quantum 2-states of the higher prequantized WZW model with cocycles in twisted K-theory. We discuss this further in 2.6.1.

A *group object* in  $\mathbf{H}$  – an  $\infty$ -group equipped with geometric structure as encoded by  $\mathbf{H}$  – may be defined to be an object equipped with a coherently homotopy-associative- hence  $A_\infty$ -multiplication, such that its 0-truncation (to a sheaf) is an ordinary group. The standard example for this is the *loop space object*  $\Omega_x X$  of any object  $X$  which is equipped with a global point,  $x : * \rightarrow X$ . This is the homotopy fiber product of the point with itself,  $\Omega_x X := *_X \times_* *$ . Here we can assume without restriction that  $X$  only has that single global point, up to equivalence, hence that  $X$  is *pointed connected*.

**Remark 2.1.12.** For  $f : B \rightarrow C$  any morphism of pointed objects in  $\mathbf{H}$ , forming successive homotopy fibers yields, due to the pasting law, a long homotopy fiber sequence in  $\mathbf{H}$  of the form

$$\dots \longrightarrow \Omega B \xrightarrow{\Omega f} \Omega C \longrightarrow A \longrightarrow B \longrightarrow C ,$$

which repeats to the left with successive loopings of the original morphism. Given an object of the form  $\mathbf{B}^n G \in \mathbf{H}$  we write  $H^n(X, G) := H(X, \mathbf{B}^n G)$  for the degree- $n$  cohomology of  $X$  with coefficients in  $G$ . Since  $\mathbf{H}(X, -)$  preserves homotopy fibers every morphism induces the expected long exact sequence in generalized nonabelian sheaf cohomology.

Another incarnation of this lifting of long sequences of homotopy groups to long sequences of homotopy types is the following.<sup>1</sup>

**Proposition 2.1.13.** *For  $f$  a morphism of pointed objects, there is for each  $n \in \mathbb{N}$  a natural equivalence*

$$\mathrm{im}_n \circ \Omega(f) \simeq \Omega \circ \mathrm{im}_{n+1}(f)$$

*between the  $n$ -image of the looping of  $f$  and the looping of the  $(n+1)$ -image of  $f$ .*

A fact that we make constant use of is that up to equivalence every  $\infty$ -group in an  $\infty$ -topos arises as the loop space object of another object, and essentially uniquely so:

**Proposition 2.1.14.** *The  $\infty$ -functor  $\Omega$  that forms loop space objects is an equivalence of  $\infty$ -categories*

$$\mathrm{Grp}(\mathbf{H}) \xrightleftharpoons[\mathbf{B}]{\Omega} \mathbf{H}_{\geq 1}^{*/}$$

*from pointed connected objects in  $\mathbf{H}$  to group objects in  $\mathbf{H}$ .*

In  $\mathbf{H} = \infty\mathrm{Grpd}$  (example 2.1.1) this is a classical fact of homotopy theory due to people like Kan, Milnor, May and Segal. In an arbitrary  $\infty$ -topos this is [31, theorem 5.1.3.6].

**Remark 2.1.15.** The inverse  $\infty$ -functor  $\mathbf{B}$  to looping usually called the *delooping* functor. The boldface here is to serve as a reminder that this is the delooping in  $\mathbf{H}$ , hence in general a *geometric* delooping which is richer (when both are comparable via a notion of geometric realization, which we come to below in 2.3) than the familiar delooping in  $\infty\mathrm{Grpd}$ , example 2.1.1, which is traditionally denoted by “ $B$ ”.

**Example 2.1.16.** In  $\mathbf{H} = \infty\mathrm{Grpd}$ , example 2.1.1, every simplicial group – a Kan complex equipped with an ordinary group structure – presents a group object, hence a (geometrically discrete)  $\infty$ -group, and up to equivalence all  $\infty$ -groups arise this way.

**Example 2.1.17.** In  $\mathbf{H} = \mathrm{Smooth}\infty\mathrm{Grpd}$ , example 2.1.2, every Lie group is canonically a smooth  $\infty$ -group

$$G \in \mathrm{LieGrp} = \mathrm{Grp}(\mathrm{SmthMfd}) \hookrightarrow \mathrm{Grp}(\mathrm{Sh}_{\infty}(\mathrm{SmthMfd})) \simeq \mathrm{Grp}(\mathrm{Smooth}\infty\mathrm{Grpd}) .$$

Accordingly a simplicial Lie group represents a smooth  $\infty$ -group. Generally, simplicial sheaves of groups represent all smooth  $\infty$ -groups, up to equivalence.

Specific examples of smooth  $\infty$ -groups that we encounter in the examples in 2.6 are the *smooth string 2-group* and the *smooth fivebrane 6-group*

$$\mathrm{String}, \mathrm{Fivebrane} \in \mathrm{Grp}(\mathrm{Smooth}\infty\mathrm{Grpd}) .$$

---

<sup>1</sup>U.S. thanks Egbert Rijke for discussion of this point.

These participate in a smooth refinement of the Whitenead tower of  $BO$  from  $\infty\text{Grpd}$  to  $\text{Smooth}\infty\text{Grpd}$ , exhibited by a diagram in  $\text{Smooth}\infty\text{Grpd}$  of the form

$$\begin{array}{ccc}
\vdots & & \vdots \\
\mathbf{B}\text{Fivebrane} & \longrightarrow & BO\langle 8 \rangle \\
\downarrow & & \downarrow \\
\mathbf{B}\text{String} & \longrightarrow & BO\langle 4 \rangle \\
\downarrow & & \downarrow \\
\mathbf{B}\text{Spin} & \longrightarrow & B\text{Spin} \\
\downarrow & & \downarrow \\
\mathbf{B}SO & \longrightarrow & BSO \\
\downarrow & & \downarrow \\
\mathbf{B}O & \xrightarrow{f} & BO
\end{array}$$

(Here the horizontal maps denote geometric realization, discussed below in 2.3, example 2.3.1.) A review of the smooth String 2-group (and of its  $\infty$ -Lie algebra, the **string** Lie 2-algebra) in a context of higher prequantum theory and string geometry is in the appendix of [33]; here we encounter this below in 2.6.1. The smooth Fivebrane 6-group was constructed in [8], a discussion in the context of higher geometric prequantum theory is in [32]; here we encounter it below in 2.6.4. For more on these matters see [D, section 5].

A group object  $G$  may admit and be equipped with further deloopings  $\mathbf{B}^k G$ , for  $k \in \mathbb{N}$ . (In terms of  $A_\infty \simeq E_1$ -structure this is a lift to  $E_k$ -structure, where  $E_k$  is the little  $k$ -cubes  $\infty$ -operad.) The higher the value of  $k$  here, the *closer to abelian* the  $\infty$ -group is:

**Definition 2.1.18.** Given a group object  $G \in \text{Grp}(\mathbf{H})$ ,

1. it is equipped with the structure of a *braided group object* if equivalently
  - $\mathbf{B}G$  is equipped with a further delooping  $\mathbf{B}^2 G$ ;
  - $\mathbf{B}G$  is itself equipped with the structure of a group object;
2. it is equipped with the structure of a *sypleptic group object* if equivalently
  - $\mathbf{B}G$  is equipped with two further deloopings  $\mathbf{B}^3 G$ ;
  - $\mathbf{B}G$  is itself equipped with the structure of a braided group object;
3. it is equipped with the structure of an *abelian group object* if it is equipped with ever higher deloopings, hence if it is an *infinite loop space object* in  $\mathbf{H}$ .

We write

$$\text{Grp}_\infty(\mathbf{H}) \rightarrow \cdots \rightarrow \text{Grp}_3(\mathbf{H}) \rightarrow \text{Grp}_2(\mathbf{H}) \rightarrow \text{Grp}_1(\mathbf{H}) := \text{Grp}(\mathbf{H})$$

for the  $\infty$ -categories of abelian  $\infty$ -groups ... sypleptic  $\infty$ -groups, braided  $\infty$ -groups and  $\infty$ -groups, respectively, with the evident forgetful functors between them.

**Example 2.1.19.** Given an abelian Lie group such as the circle group

$$U(1) \in \text{Grp}(\text{Smoothmfd}) \hookrightarrow \text{Grp}(\text{Smooth}\infty\text{Grp})$$

it is canonically an abelian  $\infty$ -group. For every  $n \in \mathbb{N}$  the  $n$ -fold geometric delooping

$$\mathbf{B}^n U(1) \in \text{Smooth}\infty\text{Grpd}$$

canonically exists and is presented under the Dold-Kan correspondence, remark 2.1.4, by the chain complex of sheaves of abelian groups concentrated on  $\underline{U}(1) = C^\infty(-, U(1))$  in degree  $n$ :

$$\mathbf{B}^n U(1) \simeq \text{DK}(\underline{U}(1)[n]) \in L_{\text{Iwhe}}[\text{SmthMfd}^{\text{op}}, \text{KanCplx}] \simeq \text{Smooth}\infty\text{Grpd}.$$

The analogous statements holds for the multiplicative Lie group  $\mathbb{C}^\times$  of invertible complex numbers. While the canonical inclusion

$$\mathbf{B}^n U(1) \hookrightarrow \mathbf{B}\mathbb{C}^\times$$

is not an equivalence in  $\text{Smooth}\infty\text{Grpd}$ , it becomes an equivalence under geometric realization  $\int : \text{Smooth}\infty\text{Grpd} \rightarrow \infty\text{Grpd} \simeq L_{\text{Iwhe}}\text{Top}$  (see 2.3 below), which maps both to

$$\int(\mathbf{B}^n U(1)) \simeq \int(\mathbf{B}^n \mathbb{C}^\times) \simeq K(\mathbb{Z}, n+1).$$

Although we happen to talk about  $U(1)$ -principal (higher) bundles throughout, using this relation all of our discussion is directly adapted to  $\mathbb{C}^\times$ -principal (higher) bundles, which is the default in some part of the literature.

**Remark 2.1.20.** For  $G \in \text{Grp}(\mathbf{H})$  an  $\infty$ -group, the geometric cohomology, remark 2.1.4, of the delooping object  $\mathbf{B}G \in \mathbf{H}$  of prop. 2.1.14 is the  $\infty$ -group cohomology of  $G$ :

$$H_{\text{grp}}(G, A) := H(\mathbf{B}G, A) := \pi_0 \mathbf{H}(\mathbf{B}G, A).$$

**Example 2.1.21.** For  $G \in \text{Grp}(\text{SmoothMfd}) \hookrightarrow \text{Grp}(\text{Smooth}\infty\text{Grpd})$  a Lie group regarded as a smooth  $\infty$ -group as in example 2.1.17, and for  $A = \mathbb{R}$  or  $A = \mathbb{Z}$  or  $A = U(1)$ , the intrinsic group cohomology of  $G$  in  $\text{Smooth}\infty\text{Grpd}$  according to remark 2.1.20 with coefficients in  $\mathbf{B}^n A$  coincides with Segal-Brylinski Lie group cohomology in degree  $n$  with these coefficients. In particular for  $G$  a compact Lie group we have

$$H_{\text{grp}}^n(G, U(1)) := \pi_0 \text{Smooth}\infty\text{Grpd}(\mathbf{B}G, \mathbf{B}^n U(1)) \simeq H^{n+1}(BG, \mathbb{Z}),$$

where on the far right we have the traditional (for instance singular) cohomology of the classifying space  $BG$ . This is a first class of examples of geometric refinement to  $\infty$ -toposes: it says that every traditional universal characteristic class  $[c] \in H^{n+1}(BG, \mathbb{Z})$  is represented by a smooth cocycle  $\nabla^0 : \mathbf{B}G \rightarrow \mathbf{B}^n U(1)$  – or equivalently, as we discuss below in 2.1.2, by a smooth  $(\mathbf{B}^{n-1}U(1))$ -principal bundle on  $\mathbf{B}G$ .

This is discussed in [D]. Below in 2.6 these smooth refinements of universal characteristic classes are seen to be the higher prequantum bundles of  $n$ -dimensional Chern-Simons type field theories.

**Remark 2.1.22.** While every object of an  $\infty$ -topos may be thought of as a higher groupoid equipped a some type of geometric structure, there is a subtlety to take note of when comparing to groupoids as traditionally used in geometry: a Lie groupoid or “differentiable stack”  $\mathcal{G}$ , as in example 2.1.2, is usually (often implicitly) regarded as a groupoid *equipped with an atlas*, namely with the canonical map  $\mathcal{G}_0 \twoheadrightarrow \mathcal{G}$  from the space of objects, regarded as a groupoid with only identity morphisms. This map is a 1-epimorphism, def. 2.1.7, hence a cover or atlas of  $\mathcal{G}$  by  $\mathcal{G}_0$ , as seen in  $\text{Smooth}\infty\text{Grpd}$ .

**Example 2.1.23.** For every group object  $G$  there is by prop. 2.1.14 an essentially unique morphism  $* \twoheadrightarrow \mathbf{B}G$ . This is a 1-epimorphism, def. 2.1.7, hence exhibits the point as an *atlas* of  $\mathbf{B}G$ . The Čech nerve of this point inclusion is a simplicial object  $(\mathbf{B}G)_n = G^{\times n} \in \mathbf{H}^{\Delta^{\text{op}}}$  in  $\mathbf{H}$ , generalizing the familiar bar construction on a group.

More generally, we say :

**Definition 2.1.24.** A simplicial object  $X_\bullet \in \mathbf{H}^{(\Delta^{\text{op}})}$  in an  $\infty$ -topos  $\mathbf{H}$  is a *pre-category* object if for all  $n \in \mathbb{N}$  the canonical projection maps

$$p_n : X_n \xrightarrow{\simeq} X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

(with  $n$  factors on the right) are equivalences, as indicated (the *Segal conditions*). These conditions imply a coherently associative partial composition operation on  $X_1$  over  $X_0$  given by

$$\circ : X_1 \times_{X_0} X_1 \xrightarrow[p_2^{-1}]{\simeq} X_2 \xrightarrow{d_1} X_1 .$$

If in a pre-category object all of  $X_1$  is invertible (up to homotopies in  $X_2$ ) under this composition operation, then it is called a *groupoid object*. We write  $\text{Grpd}(\mathbf{H}) \hookrightarrow \mathbf{H}^{\Delta^{\text{op}}}$  for the full sub- $\infty$ -category of the simplicial objects in  $\mathbf{H}$  on the groupoid objects.

This is [28, def. 6.1.2.7], here stated as in [29, section 1.1]. The following asserts that groupoid objects in this sense are indeed equivalently just objects of  $\mathbf{H}$ , but equipped with an atlas:

**Proposition 2.1.25** ( $\frac{1}{3}$ -Giraud-Rezk-Lurie axioms). *Sending 1-epimorphisms in  $\mathbf{H}$ , def. 2.1.7, to their Čech nerve simplicial objects is an equivalence of  $\infty$ -categories onto the groupoid objects in  $\mathbf{H}$ :*

$$(\mathbf{H}^{(\Delta^1)})_{\text{lepi}} \xrightarrow{\simeq} \text{Grpd}(\mathbf{H}) .$$

This is in [28, theorem 6.1.0.6, below cor. 6.2.3.5].

In order to reflect this state of affairs notationally, we stick here to the following convention on notation and terminology:

- An object  $X \in \mathbf{H} = L_{\text{the}}[C^{\text{op}}, \text{KanCplx}]$  we call an  $\infty$ -groupoid (parameterized over  $C$  or with  $C$ -geometric structure);
- a 1-epimorphism  $X_0 \twoheadrightarrow X$  we call an *atlas* for the  $\infty$ -groupoid  $X$ ;
- an object  $X_\bullet \in \text{Grpd}(\mathbf{H}) \hookrightarrow \mathbf{H}^{(\Delta^{\text{op}})}$  we call a (higher) groupoid object in  $\mathbf{H}$ ;
- the homotopy colimit over the simplicial diagram underlying a higher groupoid object, hence its *realization* as an  $\infty$ -groupoid, we indicate with the same symbol, but omitting the subscript decoration:

$$X := \lim_{\rightarrow} X_\bullet := \lim_{\rightarrow_n} X_n ;$$

- hence given a higher groupoid object denoted  $X_\bullet \in \text{Grpd}(\mathbf{H})$ , the  $\infty$ -groupoid with atlas that corresponds to it under prop. 2.1.25 we denote by  $(X_0 \twoheadrightarrow X) \in (\mathbf{H}^{(\Delta^1)})_{\text{lepi}}$ .

### 2.1.2 Higher geometric fiber bundles

If we think of a group object  $G \in \text{Grp}(\mathbf{H})$  as an  $A_\infty$ -algebra object in  $\mathbf{H}$ , then there is an evident notion of  $A_\infty$ -actions of  $G$  on any object in  $\mathbf{H}$ . This defines an  $\infty$ -category  $G\text{Act}(\mathbf{H})$  of  $G$ - $\infty$ -actions and  $G$ -equivariant maps between these. In traditional geometry, one constructs from a  $G$ -space  $V$  a universal associated bundle  $EG \times_G V \rightarrow BG$ . Analogously, in higher geometry we have a useful equivalent reformulation of  $G$ - $\infty$ -actions:



**Proposition 2.1.26.** *For  $G \in \text{Grp}(\mathbf{H})$  there is an equivalence*

$$(\mathbf{E}G) \times_G (-) : G\text{Act}(\mathbf{H}) \xrightarrow{\simeq} \mathbf{H}/\mathbf{B}G$$

*between the  $\infty$ -category of  $\infty$ -actions of  $G$  and the slice  $\infty$ -topos over the delooping  $\mathbf{B}G$ .*

This is [10, theorem 3.19, section 4.1]. In prop. 2.1.32 below we will see that this equivalence is exhibited by sending an  $\infty$ -action  $(V, \rho)$  to the corresponding *universal  $\rho$ -associated  $V$ -fiber  $\infty$ -bundle* over  $\mathbf{B}G$ . This explains our choice of notation for the  $\infty$ -functor  $(\mathbf{E}G) \times_G (-)$ .

**Definition 2.1.27.** For  $G \in \text{Grp}(\mathbf{H})$ , a  *$G$ -principal  $\infty$ -bundle* in  $\mathbf{H}$  is a map  $P \rightarrow X$  equipped with an action of  $G$  on  $P$  over  $X$  such that the map is the  $\infty$ -quotient projection  $P \rightarrow X \simeq P//G$ .

Write  $G\text{Bund}_X(\mathbf{H})$  for the  $\infty$ -category of  $G$ -principal  $\infty$ -bundles and  $G$ -equivariant maps between them fixing the base.

See [10, section 3.1] for some background discussion on the higher geometry of  $G$ -principal  $\infty$ -bundles.

**Proposition 2.1.28.** *For  $G \in \text{Grp}(\mathbf{H})$ , the map that sends a morphism in  $\mathbf{H}$  of the form  $X \rightarrow \mathbf{B}G$  to its homotopy fiber over the essentially unique point of  $\mathbf{B}G$  exhibits an equivalence with the  $\infty$ -groupoid of  $G$ -principal bundles over  $X$ :*

$$\text{fib} : \mathbf{H}(X, \mathbf{B}G) \xrightarrow{\simeq} G\text{Bund}_X(\mathbf{H}) .$$

This is [10, theorem 3.19].

**Remark 2.1.29.** Prop. 2.1.28 says that the delooping  $\mathbf{B}G \in \mathbf{H}$  is the *moduli  $\infty$ -stack* of  $G$ -principal  $\infty$ -bundles. This means that for any object  $X \in \mathbf{H}$ , maps  $\nabla^0 : X \rightarrow \mathbf{B}G$  correspond to  $G$ -principal  $\infty$ -bundles  $P \rightarrow X$  over  $X$ , and (higher) homotopies of  $\nabla^0$  correspond to (higher) gauge transformations of  $P \rightarrow X$ . Moreover, prop. 2.1.28 says that the point inclusion into  $\mathbf{B}G$  is equivalently the *universal  $G$ -principal  $\infty$ -bundle*  $\mathbf{E}G \rightarrow \mathbf{B}G$  over the moduli  $\infty$ -stack. Here and in the following we tend to denote modulating maps of  $G$ -principal bundles by “ $\nabla^0$ ”, because below in 2.4 we find that in higher prequantum geometry it is natural to regard these maps as the leftmost stage in a sequence of analogous but richer maps whose rightmost stage is a  $G$ -principal connection.

**Remark 2.1.30.** Higher geometry conceptually simplifies and strenghtens higher principal bundle theory, thereby avoiding certain difficulties which arise in ordinary principal bundle theory. For example we observe that prop. 2.1.28 has a stronger formulation, which says that, conversely, for *every*  $G$ - $\infty$ -action on some object  $V \in \mathbf{H}$ , the  $\infty$ -quotient map  $V \rightarrow V//G$  is a  $G$ -principal  $\infty$ -bundle, and that all  $G$ -principal  $\infty$ -bundles arise this way. This is a statement wildly false in ordinary geometry! It becomes true in higher geometry because homotopy colimits “correct” the quotients by non-free actions.

**Example 2.1.31.** For  $G \in \text{Grp}(\mathbf{H})$  an  $\infty$ -group and  $A \in \text{Grp}_{n+2}(\mathbf{H})$  a sufficiently deloopable  $\infty$ -group, a map of the form  $\mathbf{c} : \mathbf{B}G \rightarrow \mathbf{B}^{n+2}A$  is, by remark 2.1.20, equivalently a cocycle representing a class  $c \in H_{\text{grp}}^{n+2}(G, A)$  in the degree- $n$   $\infty$ -group cohomology of  $G$  with coefficients in  $A$ . The  $\mathbf{B}^{n+1}A$ -principal  $\infty$ -bundle

$$\begin{array}{ccc} \mathbf{B}^{n+1}A & \longrightarrow & \widehat{\mathbf{B}G} \\ & & \downarrow \text{fib}(\mathbf{c}) \\ & & \mathbf{B}G \end{array}$$

which is classified by  $\mathbf{c}$  according to prop. 2.1.28 is the delooping of the  $\infty$ -group extension

$$\begin{array}{ccc} \mathbf{B}^n A & \longrightarrow & \widehat{G} \\ & & \downarrow \Omega \text{fib}(\mathbf{c}) \\ & & G \end{array}$$

which is classified by  $c$ . Below in 2.6.4 we discuss how in higher prequantum geometry the  $\infty$ -bundles of the form  $\text{fib}(\mathbf{c})$  appear as higher prequantum bundles of higher Chern-Simons-type field theories, while  $\Omega\text{fib}(\mathbf{c})$  are those of the corresponding higher Wess-Zumino-Witten type field theories.

For  $P \rightarrow X$  a  $G$ -principal  $\infty$ -bundle and given an  $\infty$ -action  $\rho$  of  $G$  on  $V$  there is the corresponding associated  $V$ -fiber  $\infty$ -bundle  $P \times_G V \rightarrow X$  obtained by forming the  $\infty$ -quotient of the diagonal  $\infty$ -action of  $G$  on  $P \times V$ . The equivalence of prop 2.1.26 may be understood as sending  $(V, \rho)$  to the map  $V//G \rightarrow \mathbf{B}G$  which is the *universal  $\rho$ -associated  $V$ -fiber  $\infty$ -bundle*:

**Proposition 2.1.32.** *For  $P \rightarrow X$  a  $G$ -principal  $\infty$ -bundle in  $\mathbf{H}$  and for  $(V, \rho)$  a  $G$ -action, the  $\rho$ -associated  $V$ -fiber  $\infty$ -bundle  $P \times_G V \rightarrow X$  fits into a homotopy pullback square of the form*

$$\begin{array}{ccc} P \times_G V & \longrightarrow & V//G \\ \downarrow & & \downarrow \rho \\ X & \xrightarrow{\nabla^0} & \mathbf{B}G \end{array},$$

where the bottom map modulates the given  $G$ -principal bundle by prop. 2.1.28 and where the right map is the incarnation of  $\rho$  under the equivalence of prop. 2.1.26.

This is [10, prop. 4.6].

**Remark 2.1.33.** The internal hom (mapping stack)  $[(V_1, \rho_1), (V_2, \rho_2)] \in G\text{Act}(\mathbf{H})$  between two  $G$ -actions  $\rho_1$  and  $\rho_2$  on objects  $V_1, V_2 \in \mathbf{H}$  is, by prop. 2.1.26, the internal hom  $[V_1, V_2] \in \mathbf{H}$  of these two objects equipped with the induced *conjugation action*  $\rho_{\text{conj}}$  of  $G$ :

$$[(V_1, \rho_1), (V_2, \rho_2)] \simeq ([V_1, V_2], \rho_{\text{conj}}).$$

Therefore the  $G$ -homomorphisms  $V_1 \rightarrow V_2$  are those elements of  $[V_1, V_2]$  which are invariant under this conjugation action, hence the homotopy fixed points of the conjugation- $\infty$ -action. With prop. 2.1.26 and prop. 2.1.32 these are equivalently the sections of the universal  $\rho_{\text{conj}}$ -associated  $[V_1, V_2]$ -fiber  $\infty$ -bundle. Therefore by remark 2.1.10 the geometric  $\infty$ -groupoid/ $\mathbf{H}$ -object of  $G$ -equivariant maps  $V_1 \rightarrow V_2$  is

$$[(V_1, \rho_1), (V_2, \rho_2)]_{\mathbf{H}} := \prod_{\mathbf{B}G} [(V_1, \rho_1), (V_2, \rho_2)].$$

We will usually write just  $\rho$  for  $(\rho, V)$  if the space  $V$  that the action is defined on is understood. Notably with prop. 2.1.32 it follows that:

**Proposition 2.1.34.** *The space of sections of a  $V$ -fiber  $\infty$ -bundle which is  $\rho$ -associated to a principal bundle modulated by  $\nabla^0$ , is naturally equivalent to the space of maps  $\nabla^0 \rightarrow \rho$  in the slice over  $\mathbf{B}G$ :*

$$\Gamma_X(P \times_G V) \simeq \prod_{\mathbf{B}G} [\nabla^0, \rho].$$

**Example 2.1.35.** For  $G \in \text{Grp}(\text{SmoothMfd}) \hookrightarrow \text{Grp}(\text{Smooth}\infty\text{Grpd})$  a Lie group as in example 2.1.17 and for  $\rho : V \times G \rightarrow V$  an ordinary representation of  $G$  on a (vector) space  $V$ , the corresponding map  $V//G \rightarrow \mathbf{B}G$  in  $\text{Smooth}\infty\text{Grpd}$  given by prop. 2.1.26 has as its domain the object which is presented by the traditional action Lie groupoid (also called “translation Lie groupoid” etc.)

$$V//G := \left( V \times G \begin{array}{c} \xrightarrow{\rho} \\ \xleftarrow{p_1} \end{array} V \right).$$

The map itself is presented by the evident functor which forgets the  $V$ -factor. Let then  $\mathcal{U} = \{U_\alpha \rightarrow X\}_\alpha$  be a good open cover of a smooth manifold  $X$ , so that its Čech nerve groupoid  $C(\mathcal{U})$  is an equivalent resolution of  $X$ . Then a modulating map  $\nabla^0 : X \rightarrow \mathbf{B}G$  is equivalently a zig-zag

$$X \xleftarrow{\simeq} C(\mathcal{U}) \xrightarrow{g} \mathbf{B}(*, G, *)$$

hence a Čech cocycle  $\{g_{\alpha,\beta} \in C^\infty(U_{\alpha\beta}, G) | g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma}\}$ . (An introduction to these kinds of arguments is in [8].) So a dashed lift in

$$\begin{array}{ccc} & \mathbf{B}(V, G, *) & \\ \sigma \nearrow & \downarrow & \\ C(\mathcal{U}) & \xrightarrow{g} \mathbf{B}(*, G, *) & \\ \downarrow \simeq & & \\ X & & \end{array} \quad \simeq \quad \begin{array}{ccc} & V//G & \\ \sigma \nearrow & \downarrow & \\ X & \xrightarrow{\nabla^0} \mathbf{B}G & \end{array}$$

is a choice of smooth functions of the form  $\{\sigma_\alpha \in C^\infty(U_\alpha, V) | \rho(\sigma_\alpha, g_{\alpha\beta}) = \sigma_\beta\}$ . This is the traditional description terms of local data of a section of the associated  $V$ -fiber bundle  $P \times_G V$ .

**Example 2.1.36.** Every ordinary central extension of Lie groups, such as  $U(1) \rightarrow U(n) \rightarrow PU(n)$ , deloops in  $\text{Smooth}\infty\text{Grpd}$  to a long homotopy fiber sequence of the form

$$\begin{array}{ccccc} \mathbf{B}U(1) & \longrightarrow & \mathbf{B}U(n) & \longrightarrow & \mathbf{B}PU(n) \\ & & & & \downarrow \mathbf{d}d_n \\ & & & & \mathbf{B}^2U(1) \end{array},$$

where  $\mathbf{B}^2U(1)$  is as in example 2.1.19, and where the vertical map is a smooth refinement of the group 2-cocycle which classifies the extension. In this particular example, the vertical map is the universal *Dixmier-Douady class* on smooth  $PU(n)$ -principal bundles. Under prop. 2.1.26 the right part of this fiber sequence exhibits an  $\infty$ -action of the smooth circle 2-group  $\mathbf{B}U(1)$  on the moduli stack  $\mathbf{B}U(n)$ . Accordingly, for a map  $\nabla^0 : X \rightarrow \mathbf{B}^2U(1)$  modulating a  $(\mathbf{B}U(1))$ -principal 2-bundle, a section  $\sigma$  of the associated  $\mathbf{B}U(n)$ -fiber 2-bundle<sup>2</sup> over  $X$  is a dashed lift in

$$\begin{array}{ccc} & \mathbf{B}PU(n) \simeq (\mathbf{B}U(n)//\mathbf{B}U(1)) & \\ \sigma \nearrow & \downarrow \mathbf{d}d_n & \\ X & \xrightarrow{\nabla^0} \mathbf{B}^2U(1) & \end{array}.$$

The map  $\nabla^0$  here is equivalent to what is commonly called a *bundle gerbe* over  $X$ , and lifts  $\sigma$  as shown here are equivalent to what are called *bundle gerbe modules* or *rank- $n$  twisted unitary bundles* (see for instance [34]). Hence twisted bundles are sections of 2-bundles, in accord with the general remark 2.1.11.

As before for the case of ordinary sections in 1.1, the universal associated  $\mathbf{B}U(n)$ -principal bundle over  $\mathbf{B}^2U(1)$  has a differential refinement to a bundle over  $\mathbf{B}^2U(1)_{\text{conn}}$  such that dashed lifts in

$$\begin{array}{ccc} & (\mathbf{B}U(n)//\mathbf{B}U(1))_{\text{conn}} & \\ \sigma_{\text{conn}} \nearrow & \downarrow (\mathbf{d}d_n)_{\text{conn}} & \\ X & \xrightarrow{\nabla} \mathbf{B}^2U(1)_{\text{conn}} & \end{array}$$

are equivalently twisted bundles with connection.

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<sup>2</sup>This is a *Giraud  $U(n)$ -gerbe* over  $X$ , see [10, section 4.4].

## 2.2 Higher gauge groupoids

Here we discuss how to generalize, from ordinary geometry to higher geometry, the notion of a *gauge groupoid* i.e. the Lie groupoid integrating the *Atiyah Lie algebroid* associated to a smooth principal bundle. Higher gauge groupoids turn out to play a fundamental role in higher geometry. They may be viewed as the universal solution to turning an  $\mathbf{H}$ -valued automorphism  $\infty$ -group in a slice – as discussed above in 2.1.2 – into the geometric  $\infty$ -group of *bisections* of a higher groupoid. Therefore we first discuss bisections of higher groupoids in 2.2.1 and then higher gauge groupoids themselves in 2.2.2.

### 2.2.1 Bisections of higher groupoids

Traditionally, for  $\mathcal{G}_\bullet = \left( \mathcal{G}_1 \begin{smallmatrix} \xrightarrow{t} \\ \xleftarrow{s} \end{smallmatrix} \mathcal{G}_0 \right)$  a Lie groupoid, a *bisection* is defined to be a smooth function  $\sigma : \mathcal{G}_0 \rightarrow \mathcal{G}_1$  such that  $s \circ \sigma = \text{id}_{\mathcal{G}_0}$  and such that  $\phi := t \circ \sigma : \mathcal{G}_0 \rightarrow \mathcal{G}_0$  is a diffeomorphism. Just like we observed for similar cases mentioned in 1.1, a moment of reflection reveals that the group of these bisections is equivalently the following group of triangular diagrams under pasting composition

$$\mathbf{BiSect}(\mathcal{G}_\bullet) = \left\{ \begin{array}{ccc} \mathcal{G}_0 & \xrightarrow{\phi} & \mathcal{G}_0 \\ & \swarrow \sigma^{-1} & \searrow \\ & \mathcal{G} & \end{array} \begin{array}{l} \nearrow p_{\mathcal{G}} \\ \nwarrow p_{\mathcal{G}} \end{array} \right\}.$$

Hence  $\mathbf{BiSect}(\mathcal{G}_\bullet)$  is the group of automorphisms of the canonical atlas  $p_{\mathcal{G}}$  of a Lie groupoid, computed in the slice over the Lie groupoid itself.

In view of prop. 2.1.25 and remark 2.1.33 we have the following natural generalization of the notion of groupoid bisections to higher geometry.

**Definition 2.2.1.** 1. For  $G \in \text{Grp}(\mathbf{H})$ , the  $\mathbf{H}$ -valued *automorphism group* of a  $G$ -action  $\rho$  is

$$\mathbf{Aut}_{\mathbf{H}}(\rho) := \prod_{BG} \mathbf{Aut}(\rho).$$

2. For  $\mathcal{G}_\bullet \in \text{Grpd}(\mathbf{H})$  a groupoid object, def. 2.1.24, its  $\infty$ -group of *bisections* is

$$\mathbf{BiSect}(\mathcal{G}_\bullet) := \prod_{\mathcal{G}} \mathbf{Aut}(p_{\mathcal{G}}),$$

where  $p_{\mathcal{G}} : \mathcal{G}_0 \longrightarrow \mathcal{G}$  is the atlas of  $\infty$ -groupoids which corresponds to  $\mathcal{G}_\bullet$  under the equivalence of prop. 2.1.28.

The following proposition reveals a fundamental property of  $\mathbf{H}$ -valued automorphism  $\infty$ -groups in slices. As we show below, it is via this property that such  $\infty$ -groups control higher prequantum geometry.

**Proposition 2.2.2.** For  $(\sum_X E \xrightarrow{E} X) \in \mathbf{H}/_X$  an object in the slice  $\infty$ -topos of  $\mathbf{H}$  over some  $X \in \mathbf{H}$ , there is an  $\infty$ -fiber sequence in  $\mathbf{H}$  of the form

$$\Omega_E \left[ \sum_X E, X \right] \longrightarrow \mathbf{Aut}_{\mathbf{H}}(E) \longrightarrow \mathbf{Aut} \left( \sum_X E \xrightarrow{E \circ (-)} \sum_X E, X \right).$$

Here the object on the far right is regarded as pointed by  $E$  and the object on the far left is its loop space object as in prop. 2.1.14.

The above is the general abstract formalization of the basic idea schematically indicated in 1.1. This class of extensions is the archetype of all the  $\infty$ -group extensions in higher prequantum theory that we find, namely the integrated  $\infty$ -Atiyah sequence in 2.2.2 and the quantomorphism  $\infty$ -group extension in 2.5.

### 2.2.2 Higher Atiyah groupoids

A fundamental construction in the traditional theory of  $G$ -principal bundles  $P \rightarrow X$  is that of the corresponding *Atiyah Lie algebroid* [35] and the Lie groupoid which integrates it. This Lie groupoid is usually called the *gauge groupoid* of  $P$ . However, we see in 2.4 that in higher geometry there is a whole tower of higher groupoids that could go by this name. So for definiteness we stick here with the tradition of naming Lie groupoids and Lie algebroids alike and speak of the *Atiyah groupoid*  $\text{At}(P)_\bullet$ .

For  $G$  an ordinary Lie group and  $P \rightarrow X$  an ordinary  $G$ -principal bundle, the corresponding Atiyah groupoid  $\text{At}(P)_\bullet$  is the Lie groupoid whose manifold of objects is  $X$ , and whose morphisms between two points are the  $G$ -equivariant maps between the fibers of  $P$  over these points. Since a  $G$ -equivariant map between two  $G$ -torsors over the point is fixed by its image on any one point,  $\text{At}(P)_\bullet$  is usually written as on the left-hand side of

$$\begin{array}{ccc} \text{At}(P)_\bullet & \rightarrow & \text{Pair}(X)_\bullet \\ = & & = \\ \left( \begin{array}{c} (P \times P)/\text{diag } G \\ \Downarrow \\ X \end{array} \right) & & \left( \begin{array}{c} X \times X \\ \Downarrow \\ X \end{array} \right) \end{array} ,$$

where on the right-hand we display the *pair groupoid* of  $X$ . As previously discussed in 1.1, there is a conceptual simplification to this construction after the embedding into the  $\infty$ -topos  $\text{Smooth}\infty\text{Grpd}$ , example 2.1.2. Within this context the construction can be expressed in terms of the moduli stack  $\mathbf{B}G$  of  $G$ -principal bundles of prop. 2.1.29. Namely, if  $\nabla^0 : X \rightarrow \mathbf{B}G$  is the map which modulates  $P \rightarrow X$ , then:

**Proposition 2.2.3.** *The object of morphisms of  $\text{At}(P)_\bullet$  is naturally identified with the homotopy fiber product of  $\nabla^0$  with itself:*

$$\text{At}(P)_1 := (P \times P)/\text{diag } G \simeq X \times_{\mathbf{B}G} X .$$

Moreover, the canonical atlas of the Atiyah groupoid, given by the canonical inclusion  $p_{\text{At}(P)} : X \twoheadrightarrow \text{At}(P)$ , is equivalently the homotopy-colimiting cocone under the full Čech nerve of the classifying map  $\nabla^0$ :

$$\cdots \cdots \cdots X \times_{\mathbf{B}G} X \times_{\mathbf{B}G} X \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} X \times_{\mathbf{B}G} X \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} X \xrightarrow{p_{\text{At}(P)}} \left( \lim_{\rightarrow n} X \times_{\mathbf{B}G}^{n+1} \right) \simeq \text{At}(P) .$$

The full impact of this reformulation in the present context of automorphism groups in slices is seen by looking at the group of bisections, def. 2.2.1, of the Atiyah groupoid. In these terms, the above proposition 2.2.3 becomes:

**Proposition 2.2.4.** *For  $G$  a Lie group, the Atiyah groupoid  $\text{At}(P)_\bullet$  of a  $G$ -principal bundle  $P \rightarrow X$  over a smooth manifold  $X$  is the Lie groupoid which is universal with the property that its group of bisections is naturally equivalent to the group of automorphisms of the modulating map  $\nabla^0$  of  $P \rightarrow X$  (according to prop. 2.1.28) in the slice:*

$$\begin{array}{ccc} \text{BiSect}(\text{At}(P)_\bullet) & \simeq & \text{Aut}_{\mathbf{H}}(\nabla^0) \\ = & & = \\ \left\{ \begin{array}{ccc} X & \xrightarrow{\quad \phi \quad} & X \\ & \swarrow \quad \searrow & \\ p_{\text{At}(P)} & \searrow \quad \swarrow & p_{\text{At}(P)} \\ & \text{At}(P) & \end{array} \right\} & & \left\{ \begin{array}{ccc} X & \xrightarrow{\quad \phi \quad} & X \\ & \swarrow \quad \searrow & \\ \nabla^0 & \searrow \quad \swarrow & \nabla^0 \\ & \mathbf{B}G & \end{array} \right\} . \end{array}$$

Therefore, even though we have not yet introduced differential cohomology into the picture (this we turn to below in 2.3), comparison with the discussion in 1.1 shows why Atiyah groupoids are central to prequantum geometry: prequantum geometry is about automorphisms of modulating maps in slices, and the Atiyah groupoid is the universal solution to making that a group of bisections, hence making this the automorphisms of an atlas of a Lie groupoid in the slice over that Lie groupoid.

**Remark 2.2.5.** There is a more abstract formulation of this statement, which is useful in generalizing it: prop. 2.2.3 together with prop. 2.1.8 implies that after the canonical embedding of Lie groupoids into the  $\infty$ -topos  $\text{Smooth}\infty\text{Grpd}$  of example 2.1.2, the Atiyah Lie groupoid is the 1-image, in the sense of def. 2.1.7, of the modulating map  $\nabla^0$  of  $P \rightarrow X$  and its canonical atlas is the corresponding 1-image projection, hence the *first relative Postnikov stage* of  $\nabla^0$ :

$$\nabla^0 : X \xrightarrow{p_{\text{At}(P)}} \text{At}(P) \hookrightarrow \mathbf{BG} .$$

In particular we have a canonical factorizing map from  $\text{At}(P)$  to  $\mathbf{BG}$  which is a 1-monomorphism, and this implies that the components of any natural transformation from  $\nabla^0$  to itself factor through this *fully faithful* inclusion:

$$\left\{ \begin{array}{ccc} X & \xrightarrow{\phi} & X \\ & \searrow \nabla^0 & \swarrow \nabla^0 \\ & \mathbf{BG} & \end{array} \right\} \simeq \left\{ \begin{array}{ccc} X & \xrightarrow{\phi} & X \\ & \searrow p \nabla^0 & \swarrow p \nabla^0 \\ & \text{At}(P) & \\ & \downarrow & \\ & \mathbf{BG} & \end{array} \right\} .$$

This relation translates to a proof of prop. 2.2.4.

In view of these observations, it is then clear what the general definition of higher Atiyah groupoids should be: Let  $\mathbf{H}$  be an  $\infty$ -topos, let  $G \in \text{Grp}(\mathbf{H})$  be an  $\infty$ -group and let  $P \rightarrow X$  be a  $G$ -principal  $\infty$ -bundle in  $\mathbf{H}$ , as discussed above in 2.1.

**Definition 2.2.6.** The *higher Atiyah groupoid*  $\text{At}(P)_\bullet \in \text{Grpd}(\mathbf{H})$  of  $P$  is the groupoid object, def. 2.1.24, which under prop. 2.1.25 corresponds to the 1-image projection  $p_{\text{At}(P)}$

$$\nabla^0 : X \xrightarrow{p_{\text{At}(P)}} \text{At}(P) \hookrightarrow \mathbf{BG}$$

of the map  $\nabla^0$  which modulates  $P \rightarrow X$  via prop. 2.1.28.

As an illustration for the use of higher Atiyah groupoids in higher geometry, notice the following immediate rederivation and refinement to higher geometry of the classical statement in Lie groupoid theory, which says that every principal bundle arises as the source fiber of its Atiyah groupoid:

**Proposition 2.2.7.** *For  $G \in \text{Grp}(\mathbf{H})$  an  $\infty$ -group, every  $G$ -principal  $\infty$ -bundle  $P \rightarrow X$  in  $\mathbf{H}$  over an inhabited (= (-1)-connected) object  $X$  is equivalently the source-fiber of a transitive higher groupoid  $\mathcal{G}_\bullet \in \text{Grpd}(\mathbf{H})$  with vertex  $\infty$ -group  $G$  (automorphism  $\infty$ -group of any point). Here in particular we can set  $\mathcal{G}_\bullet = \text{At}(P)_\bullet$ .*

*Proof.* The outer rectangle of

$$\begin{array}{ccccc} P & \xrightarrow{\quad} & * & \xrightarrow{\quad \simeq \quad} & * \\ \downarrow & & \downarrow x & & \downarrow \\ X & \xrightarrow{\quad} & \text{At}(P) & \hookrightarrow & \mathbf{BG} \\ & \searrow \nabla^0 & & & \end{array}$$



is an  $\infty$ -pullback by prop. 2.1.28. Also the right sub-square is an  $\infty$ -pullback (for any global point  $x \in X$ ) because by  $\infty$ -pullback stability of 1-epimorphisms and 1-monomorphisms the top right morphism is a 1-monomorphism from an inhabited object to the terminal object and hence is an equivalence. Now by the pasting law for  $\infty$ -pullbacks also the left sub-square is an  $\infty$ -pullback and this exhibits  $P$  as the source fiber of  $\text{At}(P)$  over  $x \in X$ .  $\square$

Here we are interested in the following generalization to higher Atiyah groupoids of the classical facts reviewed at the beginning of this section. While this is a fairly elementary result in higher topos theory, we highlight it as a theorem since it serves as the blueprint for the differential refinement in theorem 2.5.1 below.

**Theorem 2.2.8.** *In the situation of def. 2.2.6, there is a canonical equivalence*

$$\mathbf{BiSect}(\text{At}(P)_\bullet) \simeq \mathbf{Aut}_{\mathbf{H}}(\nabla^0)$$

between the  $\infty$ -group of bisections, def. 2.2.1, of the higher Atiyah groupoid of a  $G$ -principal  $\infty$ -bundle  $P$  and the  $\mathbf{H}$ -valued automorphism  $\infty$ -group of its modulating map  $\nabla^0$ , according to prop. 2.1.28. Moreover, the  $\infty$ -group of bisections of the higher Atiyah  $\infty$ -groupoid is an  $\infty$ -group extension, example 2.1.31, of the form

$$\begin{aligned} \Omega_{\nabla^0}[X, \mathbf{BG}] &\longrightarrow \mathbf{BiSect}(\text{At}(P)_\bullet) \longrightarrow \mathbf{Aut}(X) , \\ &\simeq \\ &\mathbf{Aut}_{\mathbf{H}}(\nabla^0) \end{aligned}$$

where on the right we have the canonical forgetful map.

*Proof.* By the defining property of 1-monomorphisms and by prop. 2.2.2.  $\square$

**Remark 2.2.9.** Together with prop. 2.1.26, this theorem says that higher Atiyah groupoids are related to  $G$ -equivariant maps between the fibers of their principal  $\infty$ -bundles in just the way that one expects from the traditional situation.

Also notice that this theorem together with prop. 2.1.34 implies:

**Corollary 2.2.10.** *There is a canonical  $\infty$ -action of bisections of  $\text{At}(P)_\bullet$  on the space of sections of any associated  $V$ -fiber  $\infty$ -bundle:*

$$\mathbf{BiSect}(\text{At}(P)_\bullet) \times \mathbf{\Gamma}_X(P \times_G V) \rightarrow \mathbf{\Gamma}_X(P \times_G V) .$$

In view of theorem 2.2.8, and example 2.1.31 we may ask for a cocycle that classifies the higher Atiyah extension. This can not exist on all of  $\mathbf{Aut}(X)$ , in general, but just on the part that is in the 1-image of the projection from bisections:

**Definition 2.2.11.** For  $P \rightarrow X$  a  $G$ -principal  $\infty$ -bundle, write  $\mathbf{Aut}_P(X) \in \text{Grp}(\mathbf{H})$  for the full sub- $\infty$ -group of the automorphism  $\infty$ -group of  $X$  on those elements that have a lift to an autoequivalence of  $P$ , hence the 1-image of the right map in prop. 2.2.8:

$$\mathbf{BiSect}(\text{At}(P)_\bullet) \twoheadrightarrow \mathbf{Aut}_P(X) \hookrightarrow \mathbf{Aut}(X) .$$

**Theorem 2.2.12.** *The fiber sequence of theorem 2.2.8 extends to a long homotopy fiber sequence in  $\mathbf{H}$  of the form*

$$\Omega_{\nabla^0}[X, \mathbf{B}G] \longrightarrow \mathbf{BiSect}(\mathrm{At}(P)_\bullet) \twoheadrightarrow \mathbf{Aut}_P(X) \xrightarrow{\nabla^0 \circ (-)} \mathbf{B}(\Omega_{\nabla^0}[X, \mathbf{B}G]) .$$

Moreover, if  $G$  is a sylleptic  $\infty$ -group, def. 2.1.18, so that the rightmost object is itself naturally an  $\infty$ -group, then this naturally lifts to a long homotopy fiber sequence in  $\mathrm{Grp}(\mathbf{H})$ . In this case the delooping  $\mathbf{B}(\nabla^0 \circ (-))$  is the  $\infty$ -group cocycle (example 2.1.31) that classifies  $\mathbf{BiSect}(\mathrm{At}(P)_\bullet)$  as an  $\Omega_{\nabla^0}[X, \mathbf{B}G]$ -extension of the  $\infty$ -group  $\mathbf{Aut}_P(X)$ .

*Proof.* First consider the underlying morphisms in  $\mathbf{H}$ . By theorem 2.2.8 and by general properties of automorphisms in slices, the outer rectangle in the diagram

$$\begin{array}{ccccc} \mathbf{BiSect}(\mathrm{At}(P)_\bullet) & \twoheadrightarrow & \mathbf{Aut}_P(X) & \hookrightarrow & \mathbf{Aut}(X) \\ \downarrow & & \downarrow & & \downarrow \nabla^0 \circ (-) \\ * & \twoheadrightarrow & \mathbf{B}(\Omega_{\nabla^0}[X, \mathbf{B}G]) & \hookrightarrow & [X, \mathbf{B}G] \\ & & \searrow \vdash \nabla^0 & & \nearrow \end{array}$$

is a homotopy pullback. We form the 1-image factorization of the bottom map as indicated and observe that by homotopy pullback stability of 1-monomorphisms and 1-epimorphisms in an  $\infty$ -topos also the right and in particular also the left sub-square are then homotopy pullbacks.

Now if  $G$  is equipped with the structure of a sylleptic  $\infty$ -group, it remains to see that the vertical map in the middle lifts to a homomorphism of  $\infty$ -groups such that the left square is also a homotopy pullback in  $\mathrm{Grp}(\mathbf{H})$ .

To that end, first regard the point in the bottom left as the trivial  $\infty$ -group, and hence the bottom horizontal map uniquely as an  $\infty$ -group homomorphism. This way, by theorem 2.1.14 and by prop. 2.1.13, the top and bottom horizontal factorizations naturally lift to  $\mathrm{Grp}(\mathbf{H})$  as the looping of the 2-image factorization of the delooped horizontal morphisms. Therefore the left part of the diagram naturally lifts to a diagram of simplicial objects as shown by the solid arrows in

$$\begin{array}{ccc} \mathbf{BiSect}(\mathrm{At}(P)_\bullet)^{\times \bullet+1} & \longrightarrow & \mathbf{Aut}_P(X)^{\times \bullet+1} \\ \downarrow & & \downarrow (\nabla^0 \circ (-))_\bullet \\ *^{\times \bullet+1} & \longrightarrow & (\mathbf{B}(\Omega_{\nabla^0}[X, \mathbf{B}G]))_\bullet \end{array} ,$$

and we have to produce the dashed morphism on the right as a simplicial morphism lifting  $\nabla^0 \circ (-) = (\nabla^0 \circ (-))_0$ . Observe that each degree of the horizontal simplicial maps here is a 1-epimorphism in  $\mathbf{H}$ , because a finite product of 1-epimorphisms is still a 1-epimorphism (this follows for instance with the characterization of 1-epimorphism in prop. 2.1.8 together with the fact that  $\Delta^{\mathrm{op}}$  is a sifted  $\infty$ -category [28, prop. 5.3.1.20], so that homotopy colimits over it preserve finite products [28, lemma 5.5.8.11]). But, again by prop. 2.1.8, this induces naturally and essentially uniquely in each degree the dashed vertical morphism as the unique map between homotopy colimiting cocones under the Čech nerves of the vertical maps in this degree. Notice that here  $(\nabla^0 \circ (-))_k \simeq (\nabla^0 \circ (-))^{\times k+1}$ , necessarily, the point being that naturally implies that these components constitute a morphism of simplicial objects. Hence this diagram is degreewise a homotopy pullback in  $\mathbf{H}$ , hence is a homotopy pullback in  $\mathbf{H}^{\Delta^{\mathrm{op}}}$  and therefore finally also in  $\mathrm{Grp}(\mathbf{H})$ .  $\square$

This class of  $\infty$ -group extensions introduced in theorem 2.2.8 and theorem 2.2.12 is the source of all extensions that we consider here, and hence the source of all the fundamental extensions in traditional and in higher prequantum geometry.

For instance, a slight variation of theorem 2.2.12 adapts it to the context of differential moduli discussed below in 2.3. There it yields the central statement about the quantomorphism  $\infty$ -group extension in theorem

2.5.1. Also *higher Courant groupoids* are of this form, discussed in 2.4 below: they are intermediate between higher Atiyah groupoids and higher quantomorphism groupoids.

This fundamental unification of higher prequantum geometry via the theory of higher Atiyah groupoids is even stronger when we shift emphasis away from  $\infty$ -groups of bisections of a higher groupoid to the higher groupoid itself. Clearly, the group of bisections of a groupoid, being really the group of *global* bisections, is a global incarnation of that groupoid, and hence forgets some of its local structure. Looking back through the discussion in 1.1, we see that the main reason why one passes to groups of bisections is because these canonically *act*. For instance we saw that a prequantum operator is a tangent to a global bisection of the quantomorphism groupoid, and its action on prequantum states is inherited from the canonical action of that group of bisections.

But in fact there is a natural notion of actions of higher groupoids themselves, which refines the notion of action of their  $\infty$ -groups of bisections:

**Definition 2.2.13.** For  $\mathcal{G}_\bullet \in \text{Grpd}(\mathbf{H})$  a groupoid object and for  $p : E \rightarrow \mathcal{G}_0$  an object over  $\mathcal{G}_0$ , a *groupoid action* of  $\mathcal{G}_\bullet$  on (the space of sections of)  $E$  is another groupoid object

$$(E//\mathcal{G})_\bullet \in \text{Grpd}(\mathbf{H})$$

corresponding to an  $\infty$ -groupoid with atlas  $E \twoheadrightarrow E//\mathcal{G}$  and an  $\infty$ -pullback diagram of atlases of the form

$$\begin{array}{ccc} E & \twoheadrightarrow & E//\mathcal{G} \\ \downarrow p & & \downarrow \\ \mathcal{G}_0 & \twoheadrightarrow & \mathcal{G} \end{array} .$$

To see heuristically how such a definition indeed encodes an action, it is helpful to think of path lifting: For an element  $e \in E$  and a morphism  $(p(e) \xrightarrow{g} y) \in \mathcal{G}^{(\Delta^1)}$  in  $\mathcal{G}_\bullet$ , the  $\mathcal{G}$ -action of  $g$  on  $e$  corresponds to a lift of  $g$  to a morphism  $(e \xrightarrow{\hat{g}} \tilde{e}) \in (E//\mathcal{G})^{\Delta^1}$  in the action groupoid, which takes  $e$  to a morphism  $\tilde{e}$  sitting over  $y$ . Notice that for  $\mathcal{G} \simeq \mathbf{B}G$  the delooping groupoid of an  $\infty$ -group, this reduces to the definition of actions of  $\infty$ -groups discussed around prop. 2.1.26.

With this it is straightforward to see the canonical action of a higher Atiyah groupoid on sections of any bundle associated to its corresponding principal bundle without passing to global bisections:

**Example 2.2.14.** Given a  $G$ -principal  $\infty$ -bundle  $P \rightarrow X$  modulated by a map  $\nabla^0 : X \rightarrow \mathbf{B}G$  (prop. 2.1.28), and given a  $G$ - $\infty$ -action  $(V, \rho)$  exhibited by the universal  $V$ -bundle  $V//G \rightarrow \mathbf{B}G$  (prop. 2.1.26), recall that there is a  $\rho$ -associated  $V$ -fiber  $P \times_G V$  which fits into the homotopy pullback square described in prop. 2.1.32. The canonical  $\infty$ -action of the higher Atiyah groupoid  $\text{At}(P)_\bullet$  (def. 2.2.6) on the sections of  $P \times_X V$  is exhibited by the left square in the following pasting diagram of homotopy pullback squares:

$$\begin{array}{ccccc} P \times_G V & \longrightarrow & (P \times_G V)//\text{At}(P) & \longrightarrow & V//G \\ \downarrow & & \downarrow & & \downarrow \rho \\ X & \twoheadrightarrow & \text{At}(P) & \twoheadrightarrow & \mathbf{B}G \\ & & \searrow \nabla^0 & & \end{array} .$$

## 2.3 Higher differential geometry

The discussion of higher gauge groupoids in 2.2 makes sense in any  $\infty$ -topos and hence provides a general robust theory of higher Atiyah groupoids, 2.2.2, in all kinds of notions of geometry. However, our discussion in 2.4 below, involving the higher Heisenberg/quantomorphism groupoids and higher Courant groupoids

necessary for genuine higher prequantum geometry, requires that in the ambient  $\infty$ -topos one can give meaning to refining a  $G$ -principal bundle to a  $G$ -*principal connection*. Hence it requires to be able to refine plain (albeit geometric) cohomology described in remark 2.1.4 to *differential cohomology*. In the same fashion as indicated at the beginning of 2.1, we want to incorporate this in a flexible but robust way that allows all constructions and results to be interpreted as much as possible in various flavors of geometry such as higher/derived differential geometry, analytic geometry, supergeometry, etc. In order to achieve this, we now impose a minimum set of axioms on our ambient  $\infty$ -topos  $\mathbf{H}$ , called *cohesion* [D], that guarantees the existence of a consistent notion of differential cohomology in  $\mathbf{H}$ . Then we briefly indicate some examples and give a list of those basic constructions and results available in such a context which we use in the following chapters 2.4 and 2.5 for the formulation and study of higher prequantum geometry.

The most basic ingredient of any theory of differential cohomology is that for any coefficient object  $\mathbf{B}G \in \mathbf{H}$  there is the corresponding object  $\mathbf{B}G_{\text{disc}}$  of *discrete* coefficients equipped with a map  $u_{\mathbf{B}G} : \mathbf{B}G_{\text{disc}} \rightarrow \mathbf{B}G$ , such that a lift through this map is equivalently a *flat*  $G$ -principal connection:

$$\begin{array}{ccc} & \mathbf{B}G_{\text{disc}} & \\ \nabla_{\text{flat}} \nearrow & \downarrow u_{\mathbf{B}G} & \\ X & \xrightarrow{\nabla^0} & \mathbf{B}G \end{array} .$$

(A simple familiar example captured by this formalization is the classification of  $U(1)$ -principal bundles by degree-1 Čech cohomology with coefficients in the sheaf of  $U(1)$ -valued functions as compared to the classification of flat  $U(1)$ -principal connections by singular cohomology with coefficients in the discrete group underlying  $U(1)$ .)

Something close to this already exists in every  $\infty$ -topos  $\mathbf{H}$ : if we let

$$\flat := \text{LConst} \circ \Gamma : \mathbf{H} \rightarrow \mathbf{H}$$

be the composite of the  $\infty$ -functor  $\Gamma$  which forms global sections of  $\infty$ -stacks, with its left adjoint  $\text{LConst}$ , the  $\infty$ -functor which forms locally constant  $\infty$ -stacks, then we set

$$\mathbf{B}G_{\text{disc}} := \flat(\mathbf{B}G) \simeq \mathbf{B}(\flat G) .$$

(The symbol “ $\flat$ ” is pronounced “flat”, alluding to the relation of discrete coefficients to flat principal  $\infty$ -connections.) The counit of the adjunction  $(\text{LConst} \dashv \Gamma)$  gives the map  $u_{\mathbf{B}G}$  described above. In order to have a consistent interpretation of  $\flat G$  as the geometrically discrete version of  $G$ , it must be true that universally turning an already discrete object again into a discrete object does not change it, hence that  $u_{\flat(-)}$  is an equivalence  $\flat(\flat(-)) \xrightarrow{\simeq} \flat(-)$ . This is the *first axiom of cohesion*.

Notice that with this first axiom we may think of the image of  $\flat$  as constituting a canonical inclusion of  $\infty\text{Grpd}$  into  $\mathbf{H}$  as the geometrically discrete  $\infty$ -groupoids. In the following we freely make use of this and speak of traditional objects of homotopy theory, such as Eilenberg-MacLane spaces  $K(\mathbb{Z}, n)$ , as objects of  $\mathbf{H}$ .

Moreover, cohomology with discrete coefficients should have a consistent interpretation in terms of flat principal  $\infty$ -connections (often called *local systems of coefficients*, but better called *flat local systems of coefficients* as there are also non-flat bundles of local coefficients, see remark 2.1.11 above) and these should have a notion of (higher) *parallel transport*. In order to satisfy such design criteria, there must exist for every space  $X \in \mathbf{H}$  there exists its *fundamental  $\infty$ -groupoid* (also called *Poincaré groupoid*)  $\int X$  such that

$$\begin{array}{ccc} \text{maps} & \underbrace{X \longrightarrow \mathbf{B}G} & \text{are naturally equivalent to maps} & \underbrace{\int X \longrightarrow \mathbf{B}G} \\ \text{cocycle with discrete coefficients} & & & \text{flat parallel transport} \end{array} .$$

Technically this means that  $\flat$  has a left adjoint, or equivalently that it preserves all homotopy limits. This is the *second axiom of cohesion*.

It follows that with  $\mathbf{c} : X \rightarrow \mathbf{B}\mathbf{G}$  a map in  $\mathbf{H}$ , its image  $\int \mathbf{c} : \int X \rightarrow \int \mathbf{B}\mathbf{G}$  can be identified with a map of bare homotopy types in  $\infty\mathbf{Grpd} \simeq L_{\text{whe}}\mathbf{Top}$ , hence that  $\int$  behaves like *geometric realization* of  $\infty$ -stacks. This allows us to say what it means in  $\mathbf{H}$  to geometrically refine a cohomology class. For instance the geometric refinement  $\mathbf{c}$  in  $\mathbf{H}$  of a universal integral characteristic class  $c$  is a diagram of the form

$$\begin{array}{ccc} \mathbf{B}\mathbf{G} & \xrightarrow{\mathbf{c}} & \mathbf{B}^n U(1) \\ \downarrow \int & & \downarrow \int \\ \mathbf{B}\mathbf{G} & \xrightarrow{c} & K(\mathbb{Z}, n+1) \end{array} \quad .$$

(We see several examples of this below in 2.6.) For this interpretation to be consistent it must be true that the geometric realization of the point is contractible, and that the realization of a product is the product of the realizations. This is the *third axiom of cohesion*.

In the presence of these axioms there is a notion of non-flat principal  $\infty$ -connections, hence there is a notion of *differential cohomology* in  $\mathbf{H}$ , def. 2.3.8 below, whose coefficients are differentially refined moduli  $\infty$ -stacks which we denote by  $\mathbf{B}\mathbf{G}_{\text{conn}}$ . A special aspect of differential coefficients, discussed in detail below, is that for any object  $X \in \mathbf{H}$ , the internal hom  $[X, \mathbf{B}\mathbf{G}_{\text{conn}}]$  (the *mapping stack*) is not in general the correct moduli stack  $\mathbf{G}\mathbf{Conn}(X)$  of  $\mathbf{G}$ -principal connections on  $X$ : it has the correct global points, but not in general the expected geometric structure. One of the results presented below is that the correct differential moduli stack exists if  $\mathbf{H}$  satisfies one more condition: The  $\infty$ -functor  $\flat$  also has a right adjoint operator, to be denoted  $\sharp$ . This is the *fourth axiom of cohesion*.

**Example 2.3.1.** Our running example 2.1.2,  $\mathbf{H} = \text{Smooth}\infty\mathbf{Grpd}$ , is cohesive. Here  $\int$  sends manifolds  $X \in \text{SmthMfd} \hookrightarrow \text{Smooth}\infty\mathbf{Grpd}$  to their standard fundamental  $\infty$ -groupoid, the singular simplicial complex  $\text{Sing}(X) \in L_{\text{whe}}\mathbf{Set} \xrightarrow{L\text{Const}} \text{Smooth}\infty\mathbf{Grpd}$ , and sends moduli stacks  $\mathbf{B}\mathbf{G}$  of Lie groups and, more generally of simplicial Lie groups, to their traditional classifying spaces  $\mathbf{B}\mathbf{G} \in L_{\text{whe}}\mathbf{Top} \xrightarrow{L\text{Const}} \text{Smooth}\infty\mathbf{Grpd}$ . Generally,  $\int$  sends an  $\infty$ -stack, regarded as an  $\infty$ -functor  $\text{SmoothMfd}^{\text{op}} \rightarrow \infty\mathbf{Grpd}$  to its homotopy colimit.

Moreover, the operator  $\sharp$  on  $\text{Smooth}\infty\mathbf{Grpd}$  characterizes *concrete sheaves* on the site of smooth manifolds: the 0-truncated objects  $X \in \text{Sh}(\text{SmthMfd}) \hookrightarrow \text{Smooth}\infty\mathbf{Grpd}$  such that the unit  $X \rightarrow \sharp X$  is a 1-monomorphism. These are equivalently the *diffeological spaces*.

(If one further enhances the axioms of cohesion to those for *differential cohesion* then one can intrinsically characterize also the smooth manifolds, the orbifolds and generally the étale  $\infty$ -groupoids. This is discussed in [D, 3.5 and 3.10].)

The following table summarizes these four axioms of cohesion and their immediate interpretation in  $\mathbf{H}$ :

Axiom	$\mathbf{H}$ has a notion of...
$\flat := L\text{Const} \circ \Gamma$ is idempotent.	discrete coefficients;
There is a left adjoint $\int$ to $\flat$ .	flat principal $\infty$ -connections / flat local systems of coefficients;
$\int$ preserves finite products.	geometric realization;
There is a right adjoint $\sharp$ to $\flat$ .	differential moduli stacks.

But the point of these axioms is that they naturally imply more constructions of a differential geometric and differential cohomological nature. Some of these we turn to now.

### 2.3.1 Differential coefficients

The crucial ingredient for defining differential cohomology is the existence of universal *curvature characteristics*. Given these, differential cohomology is simply, as we see below, *curvature-twisted cohomology*, in the general sense of twisted cohomology as described in remark 2.1.11. We find universal curvature characteristics encoded in the higher homotopy fibers of the counit of the  $\flat$ -operator given by the axioms of cohesion:

**Definition 2.3.2.** For  $G \in \text{Grp}(\mathbf{H})$  an  $\infty$ -group in a cohesive  $\infty$ -topos, with delooping  $\mathbf{B}G$  (prop. 2.1.14) consider the long homotopy fiber sequence, remark 2.1.12, induced by the map  $u_{\mathbf{B}G}$ , hence consider the following pasting diagram of homotopy pullbacks:

$$\begin{array}{ccccc}
 bG & \longrightarrow & G & \longrightarrow & * \\
 \downarrow & & \downarrow \theta & & \downarrow \\
 * & \longrightarrow & b_{\text{dR}}\mathbf{B}G & \longrightarrow & b\mathbf{B}G \\
 & & \downarrow & & \downarrow u_{\mathbf{B}G} \\
 & & * & \longrightarrow & \mathbf{B}G
 \end{array}
 .$$

Here we say that

- $b_{\text{dR}}\mathbf{B}G$  is the *de Rham coefficient object* of  $\mathbf{B}G$ ;
- $\theta$  is the *Maurer-Cartan form* on  $G$ .

Moreover, if  $\mathbb{G} \in \text{Grp}_2(\mathbf{H})$  is a braided  $\infty$ -group, def. 2.1.18, then we say that the Maurer-Cartan form of its delooping group is the *universal curvature characteristic* of  $\mathbb{G}$ , denoted

$$\text{curv}_{\mathbb{G}} := \theta_{\mathbf{B}\mathbb{G}} \simeq \mathbf{B}\theta_{\mathbb{G}} : \mathbf{B}\mathbb{G} \longrightarrow \mathbf{B}b_{\text{dR}}\mathbf{B}\mathbb{G} \simeq b_{\text{dR}}\mathbf{B}^2\mathbb{G} .$$

**Example 2.3.3.** For  $G \in \text{Grp}(\text{SmthMf}) \hookrightarrow \text{Grp}(\text{Smooth}\infty\text{Grpd})$  a Lie group regarded as a smooth  $\infty$ -group as in example 2.1.17,

$$b_{\text{dR}}\mathbf{B}G = \Omega_{\text{flat}}^1(-, \mathfrak{g})$$

is given by the traditional sheaf of flat Lie-algebra valued forms and  $\theta : G \rightarrow b_{\text{dR}}\mathbf{B}G$  is, under the Yoneda embedding, the traditional Maurer-Cartan form  $\theta \in \Omega_{\text{flat}}^1(G)$ .

**Example 2.3.4.** For  $n \geq 1$  and  $\mathbb{G} = \mathbf{B}^{n-1}U(1) \in \text{Grp}(\text{Smooth}\infty\text{Grpd})$  the smooth circle  $n$ -group as in example 2.1.19, we have that

$$b_{\text{dR}}\mathbf{B}^{n+1}U(1) \simeq \mathbf{B}^n b_{\text{dR}}\mathbf{B}U(1) \simeq \text{DK}(\Omega_{\text{cl}}^1[n]) \simeq \text{DK} \left( \Omega^1 \xrightarrow{d} \dots \xrightarrow{d} \Omega^n \xrightarrow{d} \Omega_{\text{cl}}^{n+1} \right)$$

is presented under the Dold-Kan correspondence, remark 2.1.4, by the truncated and shifted de Rham complex. Moreover, the universal curvature characteristic  $\text{curv}_{\mathbf{B}^{n-1}U(1)} = \mathbf{B}^n\theta_{U(1)}$  is presented by the map which equips a  $(\mathbf{B}^{n-1}U(1))$ -principal  $n$ -bundle with a *pseudo-connection* and then sends that to the corresponding curvature in de Rham hypercohomology.

This is discussed in [8, prop. 3.2.26] and [D, section 4.4.16].

The following is a direct consequence of the axioms, but central for their interpretation in differential cohomology.

**Proposition 2.3.5.** *The universal curvature characteristic of def. 2.3.2 is the obstruction to lifting  $\mathbb{G}$ -principal  $\infty$ -bundles  $\nabla^0 : X \rightarrow \mathbf{B}\mathbb{G}$  to flat  $\infty$ -connections  $\nabla_{\text{flat}} : X \rightarrow b\mathbf{B}\mathbb{G}$ .*

Therefore:

**Definition 2.3.6.** Given a braided  $\infty$ -group  $\mathbb{G} \in \text{Grp}_2(\mathbf{H})$ , *differential  $\mathbb{G}$ -cohomology* is  $\text{curv}_{\mathbb{G}}$ -twisted cohomology, according to remark 2.1.11.



Usually, as in our applications below in 2.4 and 2.5, one chooses an object that represents a certain class of curvature twists and then restricts attention to differential cohomology obtained from just these twists. This we turn to now.

If  $\mathbf{H}$  comes equipped with differential cohesion, then it is typically desirable to consider a 0-truncated object to be denoted  $\Omega_{\text{cl}}^2(-, \mathbb{G}) \in \mathbf{H}$  which is equipped with a map

$$F_{\mathbb{G}} : \Omega_{\text{cl}}^2(-, \mathbb{G}) \longrightarrow b_{\text{dR}} \mathbf{B}^2 \mathbb{G}$$

such that for all manifolds  $\Sigma$  the map  $[\Sigma, F_{\mathbb{G}}]$  is a 1-epimorphism, def. 2.1.7, and such that  $\Omega_{\text{cl}}^2(-, \mathbb{G})$  is minimal with this property.

**Example 2.3.7.** For  $\mathbb{G} = \mathbf{B}^{n-1}U(1) \in \text{Grp}(\text{Smooth}\infty\text{Grpd})$  the smooth circle  $n$ -group as in example 2.3.4, the standard choice is to take

$$\Omega_{\text{cl}}^2(-, \mathbf{B}^{n-1}U(1)) := \Omega_{\text{cl}}^{n+1} \in \text{Sh}(\text{SmthMfd}) \hookrightarrow \text{Smooth}\infty\text{Grpd}$$

to be the ordinary sheaf of closed  $(n+1)$ -forms under the canonical inclusion into the de Rham hypercomplex presentation for  $b_{\text{dR}} \mathbf{B}^{n+1}U(1)$  from example 2.3.4. That this is a 1-epimorphism over manifolds is then equivalently the statement that every de Rham hypercohomology class on a smooth manifold has a representative by a globally defined closed differential form.

**Definition 2.3.8.** For  $\Omega_{\text{cl}}^2(-, \mathbb{G})$  a choice of curvature twists as above, we write  $\mathbf{B}\mathbb{G}_{\text{conn}} \in \mathbf{H}$  for the homotopy pullback in

$$\begin{array}{ccc} \mathbf{B}\mathbb{G}_{\text{conn}} & \xrightarrow{F_{(-)}} & \Omega_{\text{cl}}^2(-, \mathbb{G}) \\ \downarrow u_{\mathbf{B}\mathbb{G}} & & \downarrow F_{\mathbb{G}} \\ \mathbf{B}\mathbb{G} & \xrightarrow{\text{curv}_{\mathbb{G}}} & b_{\text{dR}} \mathbf{B}^2 \mathbb{G} \end{array} .$$

We say that lifts  $\nabla$  in

$$\begin{array}{ccc} & & \mathbf{B}\mathbb{G}_{\text{conn}} \\ & \nearrow \nabla & \downarrow \\ X & \xrightarrow{\nabla^0} & \mathbf{B}\mathbb{G} \end{array}$$

correspond to equipping the  $\mathbb{G}$ -principal bundle modulated by  $\nabla^0$  with a  $\mathbb{G}$ -*princial connection*. We say that lifts  $\nabla$  in

$$\begin{array}{ccc} & & \mathbf{B}\mathbb{G}_{\text{conn}} \\ & \nearrow \nabla & \downarrow F_{(-)} \\ X & \xrightarrow{\omega} & \Omega_{\text{cl}}^2(-, \mathbb{G}) \end{array}$$

are  $\mathbb{G}$ -*prequantizations* of that datum  $\omega : X \rightarrow \Omega_{\text{cl}}^2(-, \mathbb{G})$ .

**Remark 2.3.9.** A general cocycle in twisted cohomology, def. 2.1.11, with respect to the universal curvature characteristic  $\text{curv}_{\mathbb{G}}$  of def. 2.3.2 and for some twist  $\tilde{\omega}$  is given by a diagram in  $\mathbf{H}$  of the form

$$\begin{array}{ccc} X & \xrightarrow{\nabla^0} & \mathbf{B}\mathbb{G} \\ & \searrow \tilde{\omega} & \swarrow \\ & b_{\text{dR}} \mathbf{B}^2 \mathbb{G} & \end{array} .$$

This is a cocycle with coefficients in  $\mathbf{B}\mathbb{G}_{\text{conn}}$  of def. 2.3.8 if its curvature twist  $\omega$  factors through the prescribed curvature coefficients  $F_{\mathbb{G}}$  as a form datum  $\omega \in \Omega_{\text{cl}}^2(-, \mathbb{G})$ . Because in that case the universal property of the homotopy pullback identifies  $\nabla$  with the dashed morphism in the following diagram

$$\begin{array}{ccccc}
 & & \nabla^0 & & \\
 & & \curvearrowright & & \\
 X & \xrightarrow{\quad \nabla \quad} & \mathbf{B}\mathbb{G}_{\text{conn}} & \xrightarrow{\quad} & \mathbf{B}\mathbb{G} \\
 & \searrow \omega & \swarrow F_{(-)} & & \swarrow \text{curv}_{\mathbb{G}} \\
 & \Omega_{\text{cl}}^2(-, \mathbb{G}) & & & \\
 & \searrow \tilde{\omega} & \swarrow F_{\mathbb{G}} & & \\
 & & \mathbf{b}_{\text{dR}}\mathbf{B}^2\mathbb{G} & & 
 \end{array}$$

**Example 2.3.10.** For  $\mathbb{G} = \mathbf{B}^{n-1}U(1) \in \text{Grp}(\text{Smooth}\infty\text{Grpds})$  the smooth circle  $n$ -group with the standard choice of curvature twists as in example 2.3.7, the differential coefficient object  $\mathbf{B}^nU(1)_{\text{conn}}$  is presented under the Dold-Kan correspondence by the Deligne complex

$$\mathbf{B}^nU(1)_{\text{conn}} \simeq \text{DK}(\Omega^{\bullet \leq n}(-, U(1))[n]) = \text{DK}\left(\underline{U}(1) \xrightarrow{d\log} \Omega^1 \xrightarrow{d} \dots \xrightarrow{d} \Omega^n\right)$$

and the universal curvature form  $F_{(-)} : \mathbf{B}^nU(1)_{\text{con}} \rightarrow \Omega_{\text{cl}}^{n+1}$  is presented under  $\text{DK}(-)$  by the standard Deligne curvature chain map

$$\begin{array}{ccccccc}
 \underline{U}(1) & \xrightarrow{d\log} & \Omega^1 & \xrightarrow{d} & \dots & \xrightarrow{d} & \Omega^n \\
 \downarrow & & \downarrow & & \dots & & \downarrow d \\
 0 & \longrightarrow & 0 & \longrightarrow & \dots & \xrightarrow{0} & \Omega_{\text{cl}}^{n+1}
 \end{array}$$

In particular for  $X$  a smooth manifold (paracompact) and for  $\mathcal{U} = \{U_{\alpha} \rightarrow X\}_{\alpha}$  a good open cover, the chain complex

$$\text{Tot}(\mathcal{U}, \Omega^{\bullet \leq n}(-, U(1))[n])^n \xrightarrow{d_{\text{tot}}} \dots \xrightarrow{d_{\text{tot}}} \text{Tot}(\mathcal{U}, \Omega^{\bullet \leq n}(-, U(1))[n])^1 \xrightarrow{d_{\text{tot}}} \text{Tot}(\mathcal{U}, \Omega^{\bullet \leq n}(-, U(1))[n])^0_{\text{cl}}$$

is under the Dold-Kan correspondence a presentation of  $\mathbf{H}(X, \mathbf{B}^nU(1)_{\text{conn}}) \in \infty\text{Grpd}$ . Hence with respect to traditional terminology we have:

- A  $U(1)$ -principal connection in the above sense is, over a manifold  $X \in \text{SmthMfd} \hookrightarrow \text{Smooth}\infty\text{Grpd}$ , equivalently a  $U(1)$ -principal connection in the traditional sense. Over a quotient groupoid  $X//G$  it is a  $G$ -equivariant connection.
- A  $(\mathbf{B}U(1))$ -principal connection in the above sense is, over a manifold, equivalently a  $U(1)$ -bundle gerbe with connection and curving;
- A  $(\mathbf{B}^2U(1))$ -principal connection in the above sense is, over a manifold, equivalently a  $U(1)$ -bundle 2-gerbe with connection and curving and 3-form connection.

This is discussed in [8].

For the differential gauge groupoids in 2.4 below, we also need the differential coefficients which are intermediate between genuine principal  $\infty$ -connections and plain principal  $\infty$ -bundles:

**Remark 2.3.11.** If  $\mathbb{G}$  is braided, hence equipped with a further delooping, then we usually demand that the choice  $F_{\mathbb{G}}$  of curvature twists is compatible with the delooping in that we have a factorization

$$\Omega_{\text{cl}}^2(-, \mathbf{B}\mathbb{G}) \longrightarrow \mathbf{B}\Omega_{\text{cl}}^2(-, \mathbb{G}) \xrightarrow{\mathbf{B}F_{\mathbb{G}}} \mathbf{B}b_{\text{dR}}\mathbf{B}^2\mathbb{G} \xrightarrow{\simeq} b_{\text{dR}}\mathbf{B}^3\mathbb{G} .$$

$F_{\mathbf{B}\mathbb{G}}$

**Proposition 2.3.12.** For  $\mathbb{G} \in \text{Grp}_3(\mathbf{H})$  a sylleptic  $\infty$ -group, def. 2.1.18, and given a factorization of curvature twists as in remark 2.3.11, there is canonically induced a factorization

$$\mathbf{B}^2\mathbb{G}_{\text{conn}} \longrightarrow \mathbf{B}(\mathbf{B}\mathbb{G}_{\text{conn}}) \xrightarrow{\mathbf{B}u_{\mathbf{B}\mathbb{G}}} \mathbf{B}^2\mathbb{G}$$

$u_{\mathbf{B}^2\mathbb{G}}$

of the forgetful map from  $\mathbf{B}\mathbb{G}$ -principal connections to the underlying  $\mathbf{B}\mathbb{G}$ -principal bundles through the delooping of  $\mathbb{G}$ -principal connection.

*Proof.* We have a pasting diagram of  $\infty$ -pullbacks of the form

$$\begin{array}{ccc} \mathbf{B}^2\mathbb{G}_{\text{conn}} & \longrightarrow & \Omega_{\text{cl}}^2(-, \mathbf{B}\mathbb{G}) \\ \downarrow & & \downarrow \\ \mathbf{B}(\mathbf{B}\mathbb{G}_{\text{conn}}) & \longrightarrow & \mathbf{B}\Omega_{\text{cl}}^2(-, \mathbb{G}) \\ \downarrow \mathbf{B}u_{\mathbf{B}\mathbb{G}} & & \downarrow \\ \mathbf{B}\mathbf{B}\mathbb{G} & \xrightarrow{\mathbf{B}\text{curv}_{\mathbb{G}}} & \mathbf{B}b_{\text{dR}}\mathbf{B}^2\mathbb{G} \\ \downarrow \simeq & & \downarrow \simeq \\ \mathbf{B}^2\mathbb{G} & \xrightarrow{\text{curv}_{\mathbf{B}\mathbb{G}}} & b_{\text{dR}}\mathbf{B}^3\mathbb{G} \end{array}$$

$u_{\mathbf{B}^2\mathbb{G}}$   $F_{\mathbf{B}\mathbb{G}}$

□

**Definition 2.3.13.** In the situation of prop. 2.3.12 and with  $n \in \mathbb{N}$  given such that  $\mathbb{G} \in \mathbf{H}$  is  $n$ -truncated, we write  $\nabla^n := \nabla$ ,  $\nabla^{n-1}$  and  $\nabla^0$  for the three degrees of notions of  $\mathbb{G}$ -principal connections as in the diagram

$$\begin{array}{ccc} & \mathbf{B}^2\mathbb{G}_{\text{conn}} & \text{\textbf{B}\mathbb{G}-principal connection} \\ & \downarrow & \\ X & \xrightarrow{\nabla} \mathbf{B}(\mathbf{B}\mathbb{G}_{\text{conn}}) & \text{\textbf{B}\mathbb{G}-principal connection} \\ & \downarrow & \text{without top-degree connection forms} \\ X & \xrightarrow{\nabla^0} \mathbf{B}^2\mathbb{G} & \text{\textbf{B}\mathbb{G}-principal } \infty\text{-bundle} \end{array}$$

$\nabla^1$

**Example 2.3.14.** If  $\mathbb{G} = U(1) \in \text{Grp}(\text{Smooth}\infty\text{Grpd})$  in example 2.3.10 then

$$\mathbf{B}(\mathbf{B}U(1)_{\text{conn}}) \simeq \text{DK} \left( \underline{U}(1) \xrightarrow{d\log} \Omega^1 \longrightarrow 0 \right)$$

is the moduli 2-stack for what in the literature are traditionally known as  $U(1)$ -bundle gerbes with connective structure but without curving.

**Remark 2.3.15.** More generally, for sufficiently highly deloopable  $\mathbb{G}$  and compatibly chosen curvature twists in each degree, there are towers of factorizations of principal connection data

$$\begin{array}{ccc}
 & & \mathbf{B}\mathbb{G}_{\text{conn}} \\
 & \nearrow & \downarrow \\
 & \mathbf{B}\mathbb{G}_{\text{conn}^{n-1}} & \downarrow \\
 & \nearrow & \vdots \\
 X & \xrightarrow{\nabla^0} & \mathbf{B}\mathbb{G}
 \end{array}$$

### 2.3.2 Differential moduli

While differential coefficients  $\mathbf{B}\mathbb{G}_{\text{conn}}$  as discussed in 2.3.1 are the basis for any discussion of principal connections and differential cocycles, for the discussion of quantomorphism groups and (higher) Courant groupoids in (higher) prequantum geometry below in 2.4 it is crucial that we refine the construction to “concretified” differential moduli stacks. The issue here is illustrated by the following

**Example 2.3.16.** For  $n \geq 1$  let  $\Omega^n \in \text{Sh}(\text{SmthMfd}) \hookrightarrow \text{Smooth}\infty\text{Grpd}$  be the ordinary sheaf of smooth differential  $n$ -forms and let  $X \in \text{SmthMfd} \hookrightarrow \text{Smooth}\infty\text{Grpd}$  be a smooth manifold. By the  $\infty$ -Yoneda lemma, the *external* hom from  $X$  to  $\Omega^n$  is the set of smooth differential  $n$ -forms on  $X$ :

$$\mathbf{H}(X, \Omega^n) \simeq \Omega^n(X) \in \text{Set} \hookrightarrow \infty\text{Grpd}.$$

However, for various applications in differential geometry we want not just this set, but the canonical structure of a *smooth space* in particular of a *diffeological space* on this set. For instance we might have a functional on the space of  $n$ -forms on  $X$  (an action functional of a form field theory, a Hitchin function or similar) which depends smoothly on its arguments, and of which we want to form the (variational) derivative. An immediate candidate for the smooth space of  $n$ -forms on  $X$  is the *internal hom* (the *mapping stack*)

$$[X, \Omega^n] \in \text{Smooth}\infty\text{Grpd}.$$

This is an object with natural and nontrivial smooth structure (if  $X$  is not discrete) and its global points  $* \rightarrow [X, \Omega^n]$  are indeed equivalently differential  $n$ -forms on  $X$ . However,  $[X, \Omega^n]$  does not have the smooth structure which is expected of the smooth space of  $n$ -forms on  $X$ : for  $U$  a smooth test manifold, a map  $U \rightarrow [X, \Omega^n]$  is equivalently a map  $U \times X \rightarrow \Omega^n$ , which by the Yoneda lemma is equivalently a differential  $n$ -form on the product manifold  $U \times X$ . This is too much: a smoothly  $U$ -parameterized collection of differential  $n$ -forms on  $U$  should be just a *vertical* differential  $n$ -form on the bundle,  $U \times X \rightarrow U$ , hence a form on  $X \times U$  with “no legs along  $U$ ”. Notice that this issue disappears for  $n = 0$ , hence when we are dealing not with differential forms but with smooth functions: this issue is one genuine to *differential cocycles*.

**Definition 2.3.17.** Write

$$\Omega^n(X) \in \text{Sh}(\text{SmthMfd}) \hookrightarrow \text{Smooth}\infty\text{Grpd}$$

for the sheaf of such vertical  $n$ -forms.

This is the correct *moduli stack of differential  $n$ -forms on  $X$* . In this example it is easy enough to just define this by hand. But sticking to our goal of providing a flexible but robust general theory that applies broadly to higher/derived geometry and to different flavors of geometry, we observe the following abstract characterization:

**Proposition 2.3.18.** *The moduli object of differential forms, def. 2.3.17, is the 1-image, def. 2.1.6, of the unit of the  $\sharp$ -monad of smooth cohesion, example 2.3.1, applied to the internal hom of example 2.3.16:*

$$[X, \Omega^n] \longrightarrow \Omega^n(X) \hookrightarrow \sharp[X, \Omega^n] .$$

Generally we say:

**Definition 2.3.19.** Given a cohesive  $\infty$ -topos  $\mathbf{H}$  (or just a *local  $\infty$ -topos*, equipped with a  $\sharp$ -operator), we say that a 0-truncated object  $X \in \tau_{\leq 0}\mathbf{H} \hookrightarrow \mathbf{H}$  is *concrete* if  $X \rightarrow \sharp X$  is a 1-monomorphism. Moreover we say that the 1-image projection of this map is the *concretification* of  $X$ .

Hence the moduli stack  $\Omega^n(X)$  of differential forms on  $X$  is the concretification of the mapping stack  $[X, \Omega^n]$  of maps into the “differential coefficient object”  $\Omega^n$ . As we pass to differential coefficient objects that are not 0-truncated, we have to concretify the moduli stack degreewise, as shown by the following example.

**Example 2.3.20.** For  $G$  a Lie group such as  $G = U(1)$ , let  $\mathbf{BG}_{\text{conn}} \in \text{Smooth}\infty\text{Grpd}$  be the universal moduli stack for  $G$ -principal connections as in example 2.3.10. For  $X \in \text{SmoothMfd} \hookrightarrow \text{Smooth}\infty\text{Grpd}$  a smooth manifold, write

$$[X, \mathbf{BG}_{\text{conn}}] \longrightarrow \sharp_1[X, \mathbf{BG}_{\text{conn}}]$$

for the 1-image projection of its  $\sharp$ -unit. Over a test manifold  $U$  we have that the groupoid of  $U$ -plots  $U \rightarrow \sharp_1[X, \mathbf{BG}_{\text{conn}}]$  is equivalently that which has as objects the smoothly  $U$ -parameterized collections of  $G$ -principal connections on  $X$ , and has as morphisms the *discretely*  $U$ -parameterized collections of gauge transformations between these. Hence the 1-image factorization here has correctly concretified the collections of objects, but has forgotten all geometric structure on the collection of morphisms. On the other hand, a morphism of  $G$ -principal connections is just a morphism of the underlying  $G$ -principal bundles (satisfying the condition that it respects the connections) and the mapping stack  $[X, \mathbf{BG}]$  into the universal moduli for plain  $G$ -principal connections does correctly encode the geometric structure on collections of these. Therefore we can form the homotopy fiber product  $G\mathbf{Conn}(X)$  in the following diagram

$$\begin{array}{ccc} & [X, \mathbf{BG}_{\text{conn}}] & \\ & \downarrow \text{conc} & \\ & G\mathbf{Conn}(X) & \\ \swarrow & & \searrow \\ \sharp_1[X, \mathbf{BG}_{\text{conn}}] & & [X, \mathbf{BG}] \\ \searrow & & \swarrow \\ \sharp_1[X, u_{\mathbf{BG}}] & & \\ & \sharp_1[X, \mathbf{BG}] & \end{array}$$

which is the correct moduli stack  $G\mathbf{Conn}(X)$  of  $G$ -principal connections on  $X$ . Its  $U$ -plots  $U \rightarrow G\mathbf{Conn}(X)$  form the groupoid of smoothly  $U$ -parameterized collections of  $G$ -principal connections on  $X$  and of smoothly  $U$ -parameterized collections of gauge transformations between these. The vertical morphism labeled *conc* above is the one induced by the naturality of the  $\sharp_1$ -unit and the universality of the homotopy pullback. This we may call the *differential concretification* map in this case.

**Remark 2.3.21.** For  $X = *$  the point the differential concretification map here is the forgetful map from the universal moduli stack of  $G$ -principal connections to that of  $G$ -principal bundles

$$\begin{array}{ccc} [* , \mathbf{BG}_{\text{conn}}] & \simeq & \mathbf{BG}_{\text{conn}} \\ \downarrow \text{conc} & & \downarrow u_{\mathbf{BG}} \\ G\mathbf{Conn}(*) & \simeq & \mathbf{BG} \end{array} .$$

Therefore we make the following general

**Definition 2.3.22.** Let  $\mathbb{G} \in \text{Grp}_2(\mathbf{H})$  a braided  $\infty$ -group equipped with a tower of curvature twists and an induced tower of differential coefficients  $\mathbf{B}\mathbb{G}_{\text{conn}\bullet}$  as in remark 2.3.15. Then for  $X \in \mathbf{H}$  any object, the *moduli  $\infty$ -stack of  $\mathbb{G}$ -connections on  $X$*  is the iterated homotopy fiber product

$$\mathbb{G}\text{Conn}(X) := \left( \#_1[X, \mathbf{B}\mathbb{G}_{\text{conn}}] \times_{\#_1[X, \mathbf{B}\mathbb{G}_{\text{conn}^1}]} \#_2[X, \mathbf{B}\mathbb{G}_{\text{conn}^1}] \times_{\#_2[X, \mathbf{B}\mathbb{G}_{\text{conn}^2}]} \cdots \times_{\#_{n-1}[X, \mathbf{B}^n\mathbb{G}]} [X, \mathbf{B}\mathbb{G}] \right),$$

where  $\#_k(-)$  denotes the  $k$ -image factorization, def. 2.1.7 of the unit of the  $\#$ -operator.

We check that this indeed has the correct output in our running example:

**Proposition 2.3.23.** For  $\mathbb{G} = \mathbf{B}^{n-1}U(1) \in \text{Grp}(\text{Smooth}\infty\text{Grpd})$  the smooth circle  $n$ -group as in example 2.3.10 and for  $X \in \text{SmoothMfd} \hookrightarrow \text{Smooth}\infty\text{Grpd}$  a smooth manifold, the object

$$(\mathbf{B}^{n-1}U(1))\text{Conn}(X) \in \text{Smooth}\infty\text{Grpd}$$

of def. 2.3.22 is presented by the presheaf of  $n$ -groupoids which to  $U \in \text{SmoothMfd}$  assigns the  $n$ -groupoid of smoothly  $U$ -parameterized collections of Deligne cocycles on  $X$ , of smoothly  $U$ -parameterized collections of gauge transformations between these, and so on.

*Proof.* This follows by an argument generalizing the discussion in example 2.3.20. Details are in [N].  $\square$

**Remark 2.3.24.** By construction and by the universal property of homotopy limits, there is a canonical projection map

$$[X, \mathbf{B}\mathbb{G}_{\text{conn}}] \longrightarrow \mathbb{G}\text{Conn}(X)$$

from the mapping stack from  $X$  into the universal moduli stack of  $\mathbb{G}$ -principal connections of def. 2.3.8 to the concretified moduli stack of  $\mathbb{G}$ -principal connections of def. 2.3.22. We call this the *differential concretification* map. This is the higher variant of the concretification map of 0-truncated objects of def. 2.3.19.

For the discussion of the quantomorphism  $\infty$ -group extension in 2.5.1 we need the relation of the differential moduli of def. 2.3.22 to their restriction to *flat* differential cocycles:

**Definition 2.3.25.** In the situation of def. 2.3.22 we say that

$$\begin{aligned} \mathbb{G}\text{FlatConn}(X) \\ := \#_1[X, \mathbf{b}\mathbb{G}] \times_{\#_1[X, \Omega(\mathbf{B}\mathbb{G}_{\text{conn}^{n-1}})]} \#_1[X, \Omega(\mathbf{B}\mathbb{G}_{\text{conn}^{n-1}})] \times_{\#_1[X, \Omega(\mathbf{B}\mathbb{G}_{\text{conn}^{n-2}})]} \cdots \times_{\#_n[X, \Omega(\mathbf{B}\mathbb{G}_{\text{conn}^0})]} [X, \mathbb{G}] \end{aligned}$$

is the *moduli object for flat  $\mathbb{G}$ -connections on  $X$* .

**Example 2.3.26.** In the context of prop. 2.3.23 one checks that this reproduces the moduli of flat Deligne cocycles.

The crucial general abstract relation between differential moduli and flat differential moduli is now the following statement, which says that the loop space objects of the differential moduli objects are the flat differential moduli objects.

**Proposition 2.3.27.** 1. If  $\mathbb{G}$  is an abelian 0-truncated group object and if  $\int X$  is connected, then for every  $\nabla : X \rightarrow \mathbf{B}\mathbb{G}_{\text{conn}}$

$$\Omega_{\nabla}(\mathbb{G}\text{Conn}(X)) \simeq \mathbb{G}$$

2. If  $\mathbb{G}$  is not 0-truncated then

$$\Omega_*(\mathbb{G}\text{Conn}(X)) \simeq (\Omega\mathbb{G})\text{FlatConn}(X).$$



3. If moreover  $\mathbb{G}$  is a sylleptic  $\infty$ -group, then for ever  $\nabla \in \mathbb{G}\mathbf{Conn}(X)$  we have

$$\Omega_{\nabla}(\mathbb{G}\mathbf{Conn}(X)) \simeq (\Omega\mathbb{G})\mathbf{FlatConn}(X).$$

*Proof.* For the second statement observe that forming loop space objects distributes over homotopy fiber products, respects the internal hom in the second argument, commutes with the  $\sharp$ -operator (since that is right adjoint) and intertwines  $n$ -images as

$$\Omega \circ \mathrm{im}_n \simeq \mathrm{im}_{n-1} \Omega,$$

by prop. 2.1.13. The first statement follows in the same way. For the third we use that if  $\mathbb{G}$  is sylleptic then  $\mathbb{G}\mathbf{Conn}(X)$  itself is canonically a group and then the group product canonically identifies the loop space at any given point with that at the neutral element.  $\square$

**Remark 2.3.28.** The analogous construction for the not-concretified moduli stacks produces only the discrete underlying  $\infty$ -groupoid of flat higher connections, but not its cohesive structure:

$$\Omega_*[X, \mathbf{BG}_{\mathrm{conn}}] \simeq [X, \flat\mathbf{B}(\Omega\mathbb{G})] \simeq \flat((\Omega\mathbb{G})\mathbf{FlatConn}(X)).$$

## 2.4 Higher differential gauge groupoids

In 2.2 we had seen that the higher Atiyah groupoid of a  $G$ -principal  $\infty$ -bundle  $P \rightarrow X$  which is modulated by a map  $\nabla^0 : X \longrightarrow \mathbf{BG}$  to the moduli  $\infty$ -stack  $\mathbf{BG}$ , is equivalently just the 1-image projection of  $\nabla^0$ , hence its first relative Postnikov section in the ambient  $\infty$ -topos. But as such this construction does not depend on  $\mathbf{BG}$  being the moduli stack of  $G$ -principal  $\infty$ -bundles: it could be any other (moduli-) $\infty$ -stack. We now discuss how the general construction of higher Atiyah groupoids of 2.2 is naturally generalized this way to *differential* higher Atiyah groupoids which refine the traditional notion of the quantomorphism groups and Poisson bracket Lie algebras – 2.4.1 below – and of Courant algebroids – 2.4.2 below – to higher geometry.

In the following section 2.5 we then state the corresponding differential and higher analogs of the Atiyah sequence.

### 2.4.1 Higher quantomorphism- and Heisenberg-groupoids

As in the discussion in 2.3, let  $\mathbf{H}$  be a cohesive  $\infty$ -topos (such as that of smooth  $\infty$ -groupoids, example 2.3.1), let  $\mathbb{G} \in \mathrm{Grp}_2(\mathbf{H})$  be a braided  $\infty$ -group in  $\mathbf{H}$ , let  $\mathbf{BG}_{\mathrm{conn}}$  be the universal moduli  $\infty$ -stack of  $\mathbb{G}$ -principal connections.

**Definition 2.4.1.** For  $\nabla : X \rightarrow \mathbf{BG}_{\mathrm{conn}}$  the map modulating a  $\mathbb{G}$ -principal connection, the corresponding *higher quantomorphism groupoid*  $\mathrm{At}(\nabla)_\bullet \in \mathrm{Grpd}(\mathbf{H})$  or *higher contactomorphism groupoid* induced by  $\nabla$  is the corresponding higher Atiyah-groupoid according to def. 2.2.6, hence under the equivalence of prop. 2.1.25 is the  $\infty$ -groupoid with atlas which is the 1-image projection

$$X \longrightarrow \mathrm{At}(\nabla) := \mathrm{im}_1(\nabla)$$

of  $\nabla$ .

**Remark 2.4.2.** By prop. 2.2.2 the unconcretified  $\infty$ -group of bisections of the higher quantomorphism groupoid  $\mathrm{At}(\nabla)_\bullet$  of def. 2.4.1 sits in a homotopy fiber sequence of the form

$$\mathbf{BiSect}(\mathrm{At}(\nabla)_\bullet) \longrightarrow \mathbf{Aut}(X) \xrightarrow{\nabla \circ (-)} [X, \mathbf{BG}_{\mathrm{conn}}],$$

with the object on the right taken to be pointed by  $\nabla$ . But now that we are considering a differential cocycle, not just from a bundle cocycle, the same kind of reasoning as in example 2.3.16 shows that this  $\infty$ -group of bisections does have the correct global points, but does not quite have the geometric structure on these that one would typically need in applications (such as in the theorems below in 2.5). Instead, one wants the *differentially concretified* version of  $\mathbf{BiSect}(\mathrm{At}(\nabla)_\bullet)$ , along the lines of the above discussion around def. 2.3.22.

But in view of the above fiber sequence, there is a natural candidate of such differential concretification:

**Definition 2.4.3.** The *quantomorphism  $\infty$ -group* of a  $\mathbb{G}$ -principal connection  $\nabla$  is the homotopy fiber  $\mathbf{QuantMorph}(\nabla) \in \mathrm{Grp}(\mathbf{H})$  in

$$\mathbf{QuantMorph}(\nabla) \longrightarrow \mathbf{Aut}(X) \xrightarrow{\nabla \circ (-)} \mathbb{G}\mathbf{Conn}(X)$$

where the right morphism is the composite of  $\nabla \circ (-)$  with the differential concretification projection  $[X, \mathbf{B}\mathbb{G}_{\mathrm{conn}}] \longrightarrow \mathbb{G}\mathbf{Conn}(X)$  of remark 2.3.24.

**Remark 2.4.4.** The canonical forgetful map  $u_{\mathbf{B}\mathbb{G}} : \mathbf{B}\mathbb{G}_{\mathrm{conn}} \rightarrow \mathbf{B}\mathbb{G}$  induces a morphism from the higher quantomorphism groupoid to the Atiyah groupoid of the underlying  $\mathbb{G}$ -principal bundle

$$\mathrm{At}(\nabla)_\bullet \longrightarrow \mathrm{At}(\nabla^0)_\bullet$$

which is the identity on objects. This in turn induces a canonical homomorphism

$$u_{\mathbf{B}\mathbb{G}} \circ (-) : \mathbf{QuantMorph}(\nabla) \longrightarrow \mathbf{BiSect}(\mathrm{At}(\nabla)_\bullet)$$

from the quantomorphism  $\infty$ -group, def. 2.4.3, into that of bisections of the Atiyah groupoid, prop. 2.2.4. Thereby, via prop. 2.2.10, the quantomorphism  $\infty$ -group acts on the space of sections of any associated  $V$ -fiber  $\infty$ -bundle to  $\nabla^0$ . This is the *higher prequantum operator* action. It is the global version of the canonical action of the higher quantomorphism groupoid itself, in the sense of groupoid actions of def. 2.2.13, which is exhibited, in analogy with def. 2.2.14 by the left square in the following pasting diagram of  $\infty$ -pullbacks:

$$\begin{array}{ccccccc} P \times_{\mathbb{G}} V & \longrightarrow & (P \times_{\mathbb{G}} V) // \mathrm{Qu}(\nabla) & \longrightarrow & V // G \\ \downarrow & & \downarrow & & \downarrow \rho \\ X & \longrightarrow & \mathrm{Qu}(\nabla) & \xrightarrow{\hookrightarrow} & \mathbf{B}\mathbb{G}_{\mathrm{conn}} & \longrightarrow & \mathbf{B}\mathbb{G} \\ & & & \searrow & \nearrow & & \\ & & & \mathrm{At}(\nabla^0) & & & \end{array}$$

Given all of the above, we now have the following list of evident generalizations of traditional notions in prequantum theory.

**Definition 2.4.5.** Let  $\nabla : X \rightarrow \mathbf{B}\mathbb{G}_{\mathrm{conn}}$  be given, regarded as a prequantum  $\infty$ -bundle as in def. 2.3.8. Then

1. the *Hamiltonian symplectomorphism group*  $\mathbf{HamSymp}(\nabla) \in \mathrm{Grp}(\mathbf{H})$  is the sub- $\infty$ -group of the automorphisms of  $X$  which is the 1-image, def. 2.1.7, of the quantomorphisms:

$$\mathbf{QuantMorph}(\nabla) \longrightarrow \mathbf{HamSymp}(\nabla) \hookrightarrow \mathbf{Aut}(X)$$

2. for  $G \in \text{Grp}(\mathbf{H})$  an  $\infty$ -group, a *Hamiltonian  $G$ -action* on  $X$  is an  $\infty$ -group homomorphism

$$G \xrightarrow{\phi} \mathbf{HamSymp}(\nabla) \hookrightarrow \mathbf{Aut}(X) ;$$

3. an *integrated  $G$ -momentum map* is an action by quantomorphisms

$$G \xrightarrow{\hat{\phi}} \mathbf{QuantMorph}(\nabla) \hookrightarrow \mathbf{Aut}(X) ;$$

4. given a Hamiltonian  $G$ -action  $\phi$ , the corresponding *Heisenberg  $\infty$ -group*  $\mathbf{Heis}_\phi(\nabla)$  is the homotopy fiber product in

$$\begin{array}{ccc} \mathbf{Heis}_\phi(\nabla) & \longrightarrow & \mathbf{QuantMorph}(\nabla) \\ \downarrow & & \downarrow \\ G & \xrightarrow{\phi} & \mathbf{HamSymp}(\nabla) \end{array} .$$

Below in 2.6 we see examples of higher Heisenberg groups.

### 2.4.2 Higher Courant groupoids

Given a  $\mathbb{G}$ -principal  $\infty$ -connection

$$\begin{array}{ccc} & \mathbf{B}\mathbb{G}_{\text{conn}} & \\ \nearrow \nabla & \downarrow u_{\mathbf{B}\mathbb{G}} & \\ X & \xrightarrow{\nabla^0} & \mathbf{B}\mathbb{G} \end{array}$$

we have considered in 2.2 the corresponding higher Atiyah groupoid  $\text{At}(\nabla^0)_\bullet$  and in 2.4.1 the higher quantomorphism groupoid  $\text{At}(\nabla)$  equipped with a canonical map  $\text{At}(\nabla)_\bullet \longrightarrow \text{At}(\nabla^0)_\bullet$ . But in view of the towers of differential coefficients discussed in 2.3.1 this has a natural generalization to towers of higher groupoids interpolating between the higher Atiyah groupoid and the higher quantomorphism groupoid.

In particular, let  $\mathbb{G} \in \text{Grp}_3(\mathbf{H})$  a sylleptic  $\infty$ -group, def. 2.1.18, with compatibly chosen factorization of differential form coefficients and induced factorization of differential coefficients

$$\mathbf{B}^2\mathbb{G}_{\text{conn}} \longrightarrow \mathbf{B}(\mathbf{B}\mathbb{G}_{\text{conn}}) \longrightarrow \mathbf{B}^2\mathbb{G}$$

by prop. 2.3.12. Then in direct analogy with def. 2.4.1 we set:

**Definition 2.4.6.** For  $\nabla^{n-1} : X \rightarrow \mathbf{B}(\mathbf{B}\mathbb{G}_{\text{conn}})$  a  $\mathbb{G}$ -principal connection without top-degree connection data as in def. 2.3.13, we say that the corresponding *higher Courant groupoid* is the corresponding higher Atiyah groupoid  $\text{At}(\nabla^{n-1})_\bullet \in \text{Grpd}(\mathbf{H})$ , hence the groupoid object which by prop. 2.1.25 is equivalent to the  $\infty$ -groupoid with atlas given by the 1-image factorization of  $\nabla^{n-1}$

$$X \twoheadrightarrow \text{At}(\nabla^{n-1}) := \text{im}_1(\nabla^{n-1}) .$$

**Example 2.4.7.** If  $\mathbf{H} = \text{Smooth}\infty\text{Grpd}$  is the  $\infty$ -topos of smooth  $\infty$ -groupoids from example 2.3.1 and  $\mathbb{G} = \mathbf{BU}(1) \in \text{Grp}_\infty(\mathbf{H})$  is the smooth circle 2-group as in example 2.1.19 and if finally  $X \in \text{SmoothMfd} \hookrightarrow \text{Smooth}\infty\text{Grpd}$  is a smooth manifold, then by example 2.3.14 a map  $\nabla^1 : X \rightarrow \mathbf{B}(\mathbf{BU}(1)_{\text{conn}})$  is equivalently a “ $U(1)$ -bundle gerbe with connective structure but without curving” on  $X$ .

In this case the higher Courant groupoid according to def. 2.4.6 is a smooth 2-groupoid and its  $\infty$ -group of bisections  $\mathbf{BiSect}(\text{At}(\nabla^1)_\bullet)$  of def. 2.2.1 is a smooth 2-group. The points of this 2-group are equivalently pairs  $(\phi, \eta)$  consisting of a diffeomorphism  $\phi : X \xrightarrow{\cong} X$  and an equivalence of bundle gerbes with

connective structure but without curving of the form  $\eta : \phi^* \nabla^{n-1} \xrightarrow{\simeq} \nabla^{n-1}$ . A homotopy of bisections between two such pairs  $(\phi_1, \eta_1) \rightarrow (\phi_2, \eta_2)$  exists if  $\phi_1 = \phi_2$  and is then given by a higher gauge equivalence  $\kappa : \eta_1 \xrightarrow{\simeq} \eta_2$ . Moreover, with prop. 2.3.23 the smooth structure on the differentially concretified 2-group of such bisections is the expected one, where  $U$ -plots are smooth  $U$ -parameterized collections of diffeomorphisms and of bundle gerbe gauge transformations.

Precisely these smooth 2-groups have been studied in [36]. There it was shown that the Lie 2-algebras that correspond to them under Lie differentiation are the Lie 2-algebras of sections of the *Courant Lie 2-algebroid* which is traditionally associated with a bundle gerbe with connective structure. (See the citations in [36] for literature on Courant Lie 2-algebroids.) Therefore the abstractly defined smooth higher Courant groupoid  $\text{At}(\nabla^{n-1})$  according to def. 2.4.6 indeed is a Lie integration of the traditional Courant Lie 2-algebroid assigned to  $\nabla^{n-1}$ , hence is the *smooth Courant 2-groupoid*.

**Example 2.4.8.** More generally, in the situation of example 2.4.7 consider now for some  $n \geq 1$  the smooth circle  $n$ -group  $\mathbb{G} = \mathbf{B}^{n-1}U(1)$  as in example 2.1.19. Then by example 2.3.10 a map

$$\nabla^{n-1} : X \longrightarrow \mathbf{B}(\mathbf{B}^{n-1}U(1)_{\text{conn}})$$

is equivalently a Deligne cocycle on  $X$  in degree  $(n+1)$  without  $n$ -form data.

To see what the corresponding smooth higher Courant groupoid  $\text{At}(\nabla^{n-1})$  is like, consider first the local case in which  $\nabla^{n-1}$  is trivial. In this case a bisection of  $\text{At}(\nabla^{n-1})$  is readily seen to be a pair consisting of a diffeomorphism  $\phi \in \text{Diff}(X)$  together with an  $(n-1)$ -form  $H \in \Omega^{n-1}(X)$ , satisfying no further compatibility condition. This means that there is an  $L_\infty$ -algebra representing the Lie differentiation of the higher Courant groupoid  $\text{At}(\nabla^{n-1})_\bullet$  which in lowest degree is the space of sections of a bundle on  $X$  which is locally the sum  $TX \oplus \wedge^{n-1}T^*X$  of the tangent bundle with the  $(n-1)$ -form bundle. This is precisely what the sections of higher Courant Lie  $n$ -algebroids are supposed to be like, see for instance [37].

Finally, if we are given a tower of differential refinements of  $\mathbb{G}$ -principal bundles as discussed in 2.3.1

$$\begin{array}{ccc}
 & & \mathbf{B}\mathbb{G}_{\text{conn}} \\
 & \nearrow \nabla & \downarrow \\
 & & \mathbf{B}\mathbb{G}_{\text{conn}^{n-1}} \\
 & \nearrow \nabla^{n-1} & \downarrow \vdots \\
 X & \xrightarrow{\nabla^0} & \mathbf{B}\mathbb{G}
 \end{array}$$

then there is correspondingly a tower of higher gauge groupoids:

$$\begin{array}{ccccccc}
 \text{higher} & & & \text{intermediate} & & & \\
 \text{Quantomorphism} & \text{higher} & \dots & \text{differential} & \dots & \text{higher} & \\
 \text{groupoid} & \text{Courant} & & \text{higher} & & \text{Atiyah} & \\
 & \text{groupoid} & & \text{Atiyah} & & \text{groupoid} & \\
 & & & \text{groupoid} & & & 
 \end{array}$$

$$\text{At}(\nabla)_\bullet \longrightarrow \text{At}(\nabla^{n-1})_\bullet \longrightarrow \dots \longrightarrow \text{At}(\nabla^k) \longrightarrow \dots \longrightarrow \text{At}(\nabla^0)$$

The further intermediate stages appearing here seem not to correspond to anything that has already been given a name in traditional literature. We might call them *intermediate higher differential gauge groupoids*. These structures are an integral part of higher prequantum geometry.

## 2.5 Higher integrated Kostant-Souriau extensions

Conceptually, a key aspect of the traditional notion of the *Poisson bracket* Lie algebra of observables is that it constitutes a central  $\mathrm{Lie}(U(1)) = i\mathbb{R}$ -extension (over a connected manifold) of the Lie algebra of Hamiltonian vector fields – called the *Kostant-Souriau extension* (e.g [1, section 2.3]). This aspect is strengthened by the key aspect of the *quantomorphism group* extension: the corresponding  $U(1)$ -extension of the Lie group of Hamiltonian symplectomorphisms – instead of the  $\mathbb{R}$ -extension, which also is a possible Lie integration. The  $U(1)$ -phases appearing this way on top of classical Hamiltonian structures are the hallmark of (pre-)quantum geometry.

Here we discuss the refinement of these extensions to higher prequantum geometry. First we consider general quantomorphism  $\infty$ -group extensions in 2.5.1, and then the corresponding infinitesimal Poisson bracket  $L_\infty$ -algebra extensions over smooth manifolds in 2.5.2. The latter are discussed in more detail in [L].

### 2.5.1 The quantomorphism $\infty$ -group extensions

As in the discussion in 2.3.1, let  $\mathbf{H}$  be a cohesive  $\infty$ -topos (such as  $\mathrm{Smooth}\infty\mathrm{Grpd}$  of example 2.3.1), let  $\mathbb{G} \in \mathrm{Grp}_2(\mathbf{H})$  a braided  $\infty$ -group, def. 2.1.18, let  $X \in \mathbf{H}$  any object, let  $\omega : X \rightarrow \Omega_{\mathrm{cl}}^2(-, \mathbb{G})$  be a flat differential form datum and let  $\nabla : X \rightarrow \mathbf{B}\mathbb{G}_{\mathrm{conn}}$  a  $\mathbb{G}$ -prequantization of it. Then we have the following characterization of the corresponding quantomorphism  $\infty$ -group of def. 2.4.3.

**Theorem 2.5.1.** *There is a long homotopy fiber sequence in  $\mathrm{Grp}(\mathbf{H})$  of the form*

- if  $\mathbb{G}$  is 0-truncated:

$$\mathbb{G} \longrightarrow \mathbf{QuantMorph}(\nabla) \longrightarrow \mathbf{HamSymp}(\nabla) \xrightarrow{\nabla \circ (-)} \mathbf{B}(\mathbb{G}\mathbf{ConstFunct}(X))$$

- otherwise:

$$(\Omega\mathbb{G})\mathbf{FlatConn}(X) \longrightarrow \mathbf{QuantMorph}(\nabla) \longrightarrow \mathbf{HamSymp}(\nabla) \xrightarrow{\nabla \circ (-)} \mathbf{B}((\Omega\mathbb{G})\mathbf{FlatConn}(\nabla)) ,$$

which hence exhibits the quantomorphism group  $\mathbf{QuantMorph}(\nabla) \in \mathrm{Grp}(\mathbf{H})$  as an  $\infty$ -group extension, example 2.1.31, of the  $\infty$ -group of Hamiltonian symplectomorphisms, def. 2.4.5, by the differential moduli of flat  $\Omega\mathbb{G}$ -principal connections on  $X$ , def. 2.3.25, classified by an  $\infty$ -group cocycle which is given by postcomposition with  $\nabla$  itself.

*Proof.* This is an immediate variant, under the differential concretification of def. 2.3.22, of the higher Atiyah sequence of theorem 2.2.12: Consider the natural 1-image factorization of the horizontal maps in the defining  $\infty$ -pullback of def. 2.4.3:

$$\begin{array}{ccccc} \mathbf{QuanMorph}(\nabla) & \longrightarrow & \mathbf{HamSymp}(\nabla) & \hookrightarrow & \mathbf{Aut}(X) \\ \downarrow & & \downarrow \nabla \circ (-) & & \downarrow \nabla \circ (-) \\ * & \longrightarrow & \mathbf{B}(\Omega_{\nabla}(\mathbb{G}\mathbf{Conn}(X))) & \hookrightarrow & \mathbb{G}\mathbf{Conn}(X) \\ & & \searrow \vdash \nabla & & \end{array}$$

By homotopy pullback stability of both 1-epimorphisms and 1-monomorphisms and by essential uniqueness of 1-image factorizations this is a pasting diagram of homotopy pullback squares. The claim then follows with prop. 2.3.27 as in the proof of theorem 2.2.12.  $\square$

The analogous statement also holds for Heisenberg  $\infty$ -groups:

**Corollary 2.5.2.** *If  $\phi : G \rightarrow \mathbf{HamSymp}(\nabla) \hookrightarrow \mathbf{Aut}(X)$  is any Hamiltonian  $G$ -action, def. 2.4.5, then the corresponding Heisenberg  $\infty$ -group sits in the  $\infty$ -fiber sequence*

$$(\Omega\mathbb{G})\mathbf{FlatConn}(X) \longrightarrow \mathbf{Heis}_\phi(\nabla) \longrightarrow G \xrightarrow{\nabla \circ (-)} \mathbf{B}((\Omega\mathbb{G})\mathbf{FlatConn}(\nabla)) ,$$

*Proof.* By the pasting law for homotopy pullbacks. □

**Example 2.5.3.** For  $\mathbb{G} = U(1) \in \mathbf{Grp}(\mathbf{Smooth}\infty\mathbf{Grpd})$  the smooth circle group as in example 2.3.4 for  $n = 1$ , and for  $X \in \mathbf{SmoothMfd} \hookrightarrow \mathbf{Smooth}\infty\mathbf{Grpd}$  a connected smooth manifold, theorem 2.5.1 reproduces the traditional quantomorphism group as a  $U(1)$ -extension of the traditional group of Hamiltonian symplectomorphisms, as discussed for instance in [6, 7].

In order to put the higher generalizations of the quantomorphism extensions into this context, we notice the following basic fact.

**Proposition 2.5.4.** *For  $\mathbb{G} = \mathbf{BU}(1) \in \mathbf{Grp}(\mathbf{Smooth}\infty\mathbf{Grpd})$  the smooth circle 2-group as in example 2.3.4 for  $n = 2$ , consider  $X \in \mathbf{SmoothMfd} \hookrightarrow \mathbf{Smooth}\infty\mathbf{Grpd}$  a connected and simply connected smooth manifold. Then from prop. 2.3.23 and example 2.3.26 one obtains an equivalence of smooth group stacks*

$$U(1)\mathbf{FlatConn}(X) \simeq \mathbf{BU}(1) .$$

*Generally, for  $n \geq 1$  and for  $\mathbb{G} = \mathbf{B}^n U(1) \in \mathbf{Grp}(\mathbf{Smooth}\infty\mathbf{Grpd})$  the smooth circle  $(n + 1)$ -group as in example 2.3.4, there is for  $X$  an  $n$ -connected smooth manifold an equivalence of smooth  $\infty$ -groups*

$$(\mathbf{B}^{n-1}U(1))\mathbf{FlatConn}(X) \simeq \mathbf{B}^n U(1) .$$

*Proof.* We use the description of  $U(1)\mathbf{FlatConn}(X)$  given by prop. 2.3.23 and example 2.3.26. First notice then that on a simply connected manifold there is up to equivalence just a single flat connection, hence  $U(1)\mathbf{FlatConn}(X)$  is pointed connected. Moreover, an auto-gauge transformation from that single flat connection (any one) to itself is a  $U(1)$ -valued function which is *constant on  $X$* . But therefore by prop. 2.3.23 the  $U$ -plots of the first homotopy sheaf of  $U(1)\mathbf{FlatConn}(X)$  are smoothly  $U$ -parameterized collections of constant  $U(1)$ -valued functions on  $X$ , hence are smoothly  $U$ -parameterized collections of elements in  $U(1)$ , hence are smooth  $U(1)$ -valued functions on  $U$ . These are, by definition, equivalently the  $U$ -plots of automorphisms of the point in  $\mathbf{BU}(1)$ . □

The other cases work analogously.

**Remark 2.5.5.** Therefore in the situation of prop. 2.5.4 the quantomorphism  $\infty$ -group is a smooth 2-group extension by the circle 2-group  $\mathbf{BU}(1)$ . The archetypical example of  $\mathbf{BU}(1)$ -extensions is the smooth *String 2-group*, example 2.1.17. Indeed, this occurs as the Heisenberg 2-group extension of the WZW sigma-model regarded as a local 2-dimensional quantum field theory. This we turn to in 2.6.1 below.

## 2.5.2 The higher Poisson bracket $L_\infty$ -extensions

Above in 2.5.1 we have established the quantomorphism  $\infty$ -group extension in higher prequantum geometry in generality. Here we now consider in more detail the archetypical special case of higher prequantum differential geometry over smooth manifolds. We consider the  $\infty$ -Lie algebra  $\mathfrak{Poisson}(X, \omega)$  which is obtained from a smooth quantomorphism  $n$ -group  $\mathbf{QuantMorph}(\nabla)$  under Lie differentiation. We show that this is presented by the higher Poisson bracket  $L_\infty$ -algebras proposed in [16] and we establish the  $L_\infty$ -algebra analog of the quantomorphism extension by giving an explicit  $L_\infty$ -cocycle which classifies the Poisson bracket  $L_\infty$ -algebra as an  $L_\infty$ -extension of the Lie algebra of Hamiltonian vector fields. For  $n = 1$  this reproduces the traditional Kostant-Souriau central extension. For  $n = 2$  it recovers the higher Kostant-Souriau central extension discussed in [39, Sec. 9], where higher prequantization was first developed in this degree.

**Remark 2.5.6.** With respect to the above general discussion we consider now the ambient  $\infty$ -topos to be that of smooth groupoids  $\mathbf{H} = \text{Smooth}\infty\text{Grpds} = \text{Sh}_\infty(\text{SmthMfd})$  from example 2.1.2 and consider base space to be a smooth manifold (under the  $\infty$ -Yoneda embedding)

$$X \in \text{SmthMfd} \hookrightarrow \text{Smooth}\infty\text{Grpd}.$$

But the discussion in this section here is independent of higher topos theory and in itself just involves traditional differential geometry and  $L_\infty$ -algebra theory.

In [16] higher prequantum geometry in the context of differential geometry over smooth manifolds was conceptualized as follows.

**Definition 2.5.7.** Let  $n \in \mathbb{N}$ ,  $n \geq 1$ . A pair  $(X, \omega)$  consisting of a smooth manifold  $X$  and a closed  $(n+1)$ -form  $\omega \in \Omega_{\text{cl}}^{n+1}(X)$  is called *pre- $n$ -plectic manifold*. If  $\omega$  is non-degenerate in that the map  $\iota_{(-)}\omega : T_x X \rightarrow \wedge^n T_x^* X$  is injective for all  $x \in X$ , then we say that the pre- $n$ -plectic manifold is an  *$n$ -plectic manifold*.

For  $v \in \Gamma(TX)$  and  $h \in \Omega^{n-1}(X)$  such that  $\iota_v \omega + dh = 0$  we say that  $v$  is a *Hamiltonian vector field* (with respect to the pre- $n$ -plectic structure  $\omega$ ) and that  $h$  is a *Hamiltonian form* for  $v$ . We write

$$\mathcal{X}_{\text{Ham}} \hookrightarrow \Gamma(TX)$$

for the sub-Lie algebra of Hamiltonian vector fields.

**Definition 2.5.8.** For  $(X, \omega)$  a pre- $n$ -plectic manifold, write

$$L_\infty(X, \omega) \in L_\infty\text{Alg}$$

for the  $L_\infty$ -algebra

- whose underlying chain complex is the modified de Rham complex

$$\Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n-2}(X) \xrightarrow{(0,d)} \Omega_{\text{Ham}}^{n-1}(X),$$

where

$$\Omega_{\text{Ham}}^{n-1}(X) := \{(v, h) \in \Gamma(TX) \oplus \Omega^{n-1}(X) \mid \iota_v \omega + dh = 0\},$$

- whose only non-vanishing brackets are on tuples of elements  $(v_i, h_i) \in \Omega_{\text{Ham}}^{n-1}(X, \omega)$  in degree 0, where the binary bracket is

$$[(v_1, h_1), (v_2, h_2)] := ([v_1, v_2], \iota_{v_1 \wedge v_2} \omega)$$

and where the brackets of arity  $k \geq 3$  are

$$[(v_1, h_1), \dots, (v_k, h_k)] := (-1)^{\lfloor \frac{k-1}{2} \rfloor} \iota_{v_1 \wedge \dots \wedge v_n} \omega.$$

We call  $L_\infty(X, \omega)$  the *de Rham model* for the high Poisson bracket of local observables.

For  $n$ -plectic manifolds this  $L_\infty$ -algebra is isomorphic to the one given in [16, theorem 5.2]. For the more general pre- $n$ -plectic case this appears as [40, theorem 4.7].

**Example 2.5.9.** For  $n = 1$  and non-degenerate  $\omega$ , hence for  $(X, \omega)$  an ordinary symplectic manifold,  $L_\infty(X, \omega)$  is the traditional Poisson bracket Lie algebra on the space of smooth functions  $C^\infty(X)$  on  $X$ . If  $X$  is thought of as the phase space of a physical system, then each point of it corresponds to a configuration or trajectory of the system, and a function on  $X$  hence assigns a value to each such configuration. One thinks of this value as a physically observable property of the given configuration, for instance its energy. In this way functions on  $X$  are “observables” for  $n = 1$ .



**Remark 2.5.10.** For  $n > 1$ , as the name suggests, the  $L_\infty$ -algebra given in def. 2.5.8 has an interpretation as a higher Lie algebra of “local observables” within the context of an  $n$ -dimensional local field theory. (This was demonstrated, for example, in [41] for the  $n = 2$  case.) But notice that the usual rules of homotopy theory imply a nuanced notion of what “a local observable” is. In particular, we have the following observations:

1. On the one hand there are the principal homotopy-invariants of the chain complex of observables  $L_\infty(X, \omega)$ , namely its homology groups  $H_\bullet(L_\infty(X, \omega))$ . The traditional notion of a “local observable” is an element in one of these homology groups. This means that a degree 0 local observable in the “strict sense” is a Hamiltonian  $(n-1)$ -form  $j$  on  $X$  modulo exact forms. This is a familiar structure in quantum field theory: such a *local current up to gauge transformation* is something that when integrated over a (closed) spatial slice of spacetime (hence: when transgressed to codimension 1) produces a 0-form observable as in example 2.5.9. This is called the *total charge* of the current.
2. On the other hand, it is a crucial fact in homotopy theory that a homotopy type is not, in general, faithfully encoded by its homotopy groups (here: homology groups). Therefore a “local observable”, in a more accurate and less restricted sense, is *any element* of  $L_\infty(X, \omega)$ . Moreover, one should also remember the relation between these elements under the notion of homotopy in the complex, hence under the chain maps.

For instance, the de Donder-Weyl Hamiltonian  $H_{\text{DW}}$  of multisymplectic geometry (see [38, around (4)] for a review) is a smooth function on the  $n$ -plectic manifold  $(X, \omega)$  which characterizes  $n$ -tuples of Hamiltonian vector fields  $(v_1, \dots, v_n)$  tangent to the classical solution hypersurfaces by the equation

$$dH_{\text{DW}} = [v_1, \dots, v_n]_{L_\infty(X, \omega)}.$$

This is the “localized equation of motion” which de-transgresses the traditional equation of motion in symplectic geometry. In view of def. 2.5.8 this equation exhibits  $H_{\text{DW}}$  as a homotopy in the  $\infty$ -groupoid of local observables that connects the  $n$ -ary  $L_\infty$ -bracket of the  $n$ -tuple of Hamiltonian vector fields to the origin. So the deDonder-Weyl Hamiltonian is *not a local observable in the strict sense*. Indeed, it crucially is not closed, in general, and hence does not represent an element in a homology group of  $L_\infty(X, \omega)$ . It is, however, an observable in a homotopy-theoretic sense, since it gives a *homotopy* between “strict” observables.

For the case that  $(X, \omega)$  admits a  $(\mathbf{B}^{n-1}U(1))$ -prequantization according to def. 2.3.8 and example 2.3.10, we now give a *strictification* of  $L_\infty(X, \omega)$ , hence an equivalent  $L_\infty$ -algebra with only unary and binary brackets, hence an equivalent *dg-Lie algebra*. This strict presentation is a subalgebra of the following class of dg-Lie algebras.

**Definition 2.5.11.** For  $X$  a smooth manifold and  $\mathcal{U} = \{U_\alpha \rightarrow X\}_\alpha$  be a good open cover, write

$$\Gamma(TX) \ltimes (\text{Tot}(\mathcal{U}, \Omega^\bullet)[n-1])^{\leq n-1} \in \text{dgLieAlg} \quad (1)$$

for the semidirect product dg-Lie algebra of the Lie algebra of vector fields on  $X$  acting by Lie derivation on the shifted and truncated total Čech-de Rham complex relative to  $\mathcal{U}$ .

**Example 2.5.12.** Let  $\mathcal{U}$  be the trivial cover. If  $n = 1$  then def. 2.5.11 reduces to the semidirect product Lie algebra  $\Gamma(TX) \ltimes C^\infty(X)$  of vector fields acting on smooth functions by differentiation. For  $n > 1$ , the underlying chain complex of def. 2.5.11 is

$$\begin{array}{ccccccc} C^\infty(X) & \xrightarrow{d} & \Omega^1(X) & \xrightarrow{d} & \dots & \xrightarrow{d} & \Omega^{n-1}(X) \\ & & & & & & \oplus \\ & & & & & & \Gamma(TX) \end{array}$$

and  $\Gamma(TX)$  acts on all degrees by Lie differentiation.

If  $\mathcal{U}$  is a general cover, then the elements in degree 0, for instance, involve  $(n-1)$ -forms on single patches,  $(n-2)$ -forms on double intersections, etc. A vector field in  $\Gamma(TX)$  still acts by Lie differentiation after restricting it to a given  $k$ -fold intersection.

**Definition 2.5.13.** Let  $(X, \omega)$  be a pre- $n$ -plectic manifold such that  $\omega$  is integral. Let  $\mathcal{U} = \{U_\alpha \rightarrow X\}_\alpha$  be a good open cover and let  $A = \{A_\alpha, A_{\alpha\beta}, A_{\alpha\beta\gamma}, \dots\}$  be a Čech-Deligne cocycle relative  $\mathcal{U}$  with curvature  $\omega$ . We denote by

$$\mathrm{dgLie}(X, A) \in \mathrm{dgLieAlg} \hookrightarrow L_\infty \mathrm{Alg}$$

the dg Lie subalgebra of (1) whose degree-0 components  $(v, b)$  satisfy

$$\mathcal{L}_v A = d_{\mathrm{tot}} b.$$

We call this the *strict model of the higher Poisson bracket Lie algebra of local observables* of  $(X, \omega)$ .

**Remark 2.5.14.** We indicate in [L, section 4] how the Lie differentiation of **QuantMorph**( $\nabla$ ) yields the dg-Lie algebra  $\mathrm{dgLie}(X, \nabla)$  of def. 2.5.13.

**Proposition 2.5.15.** *If  $(X, \omega)$  is a pre- $n$ -plectic manifold and  $A \in \mathrm{Tot}(\mathcal{U}, \Omega^\bullet(-, U(1)))$  is a Čech-Deligne cocycle over a cover  $\mathcal{U}$  of  $X$  with curvature  $\omega$ , then there is a  $L_\infty$ -quasi-isomorphism*

$$\mathrm{dgLie}(X, A) \xrightarrow{\cong} L_\infty(X, \omega)$$

*between the de Rham model (def. 2.5.8) and the strict model (def. 2.5.13) of the higher Lie algebra of local observables.*

**Remark 2.5.16.** In particular this means that  $\mathrm{dgLie}(X, A)$  is independent, up to equivalence, of the chosen prequantization  $A$  of  $\omega$ .

The  $\infty$ -category  $L_{\mathrm{we}}(L_\infty \mathrm{Alg})$  of  $L_\infty$ -algebras can be obtained by the simplicial localization of  $L_\infty \mathrm{Alg}$  at the  $L_\infty$ -quasi-isomorphisms. Prop. 2.5.15 suggests that it is useful to adopt an intrinsically homotopy-theoretic notation:

**Definition 2.5.17.** For  $(X, \omega)$  a prequantizable pre- $n$ -plectic manifold, we denote by

$$\mathfrak{Poisson}(X, \omega) \in L_{\mathrm{we}}(L_\infty \mathrm{Alg})$$

the homotopy type of the  $L_\infty$ -algebra which is presented via prop. 2.5.15. We call this the *Poisson bracket  $\infty$ -Lie algebra of local observables* of the pre- $n$ -plectic manifold  $(X, \omega)$ .

The following theorem now characterizes  $\mathfrak{Poisson}(X, \omega)$  more abstractly/more intrinsically. This is the analog under Lie differentiation of theorem 2.5.1 above.

**Theorem 2.5.18** (higher Kostant-Souriau extension). *For  $(X, \omega)$  a pre- $n$ -plectic manifold there is a long homotopy fiber sequence of  $L_\infty$ -algebras of the form*

$$\mathbf{H}(X, \mathfrak{b}\mathbf{B}^{n-1}\mathbb{R}) \longrightarrow \mathfrak{Poisson}(X, \omega) \longrightarrow \mathcal{X}_{\mathrm{Ham}}(X, \omega) \xrightarrow{\iota_{(-)}\omega} \mathbf{BH}(X, \mathfrak{b}\mathbf{B}^{n-1}\mathbb{R}) ,$$

where

- $\mathbf{H}(X, \mathfrak{b}\mathbf{B}^{n-1}\mathbb{R})$  is presented by the truncated de Rham complex of  $X$  for degree- $n$  de Rham cohomology regarded as an abelian  $L_\infty$ -algebra;
- $\mathcal{X}_{\mathrm{Ham}}(X, \omega)$  is the ordinary Lie algebra of vector fields on the  $\omega$ -Hamiltonian vector fields;
- $\iota_{(-)}\omega$  is given on the model  $L_\infty(X, \omega)$  by the  $L_\infty$ -homomorphism whose  $k$ -ary components are given by contracting skew tensor products of  $k$  vector fields with  $\omega$ .

This is discussed in detail in [L].

**Remark 2.5.19.** Theorem 2.5.18 says that  $\iota_{(-)}\omega$  is the  $L_\infty$ -cocycle which classifies the Poisson bracket  $L_\infty$ -algebra of local observables as an  $L_\infty$ -extension of the Lie algebra of Hamiltonian vector fields. This cocycle is an  $L_\infty$ -generalization of the traditional *Heisenberg cocycle* classifying the traditional Heisenberg group extension.

**Remark 2.5.20.** Together with prop. 2.5.15, theorem 2.5.18 identifies the  $L_\infty$  algebra of def. 2.5.8 as a natural higher analogue of the Poisson bracket Lie bracket of ordinary symplectic geometry. Other proposals in the literature for what a higher analog of the Poisson bracket Lie algebra should be as one generalizes from symplectic forms to higher differential forms can be found in [21]. These other definition are not manifestly equivalent to def. 2.5.8 and it seems unlikely that they are equivalent to it in a more subtle way.

Finally, the image under Lie differentiation of the Heisenbger  $\infty$ -groups as in def. 2.4.5 is

**Definition 2.5.21.** For  $\mathfrak{g}$  an  $L_\infty$ -algebra and  $\rho : \mathfrak{g} \rightarrow \mathcal{X}_{\text{Ham}}(X)$  a Hamiltonian action the homotopy fiber of  $\iota_{(-)}\omega \circ \rho$ , hence the homotopy pullback of  $\mathfrak{Poisson}(X, \omega)$  along  $\rho$ , is the *Heisenberg  $L_\infty$ -extension*  $\mathfrak{Heis}_\rho(\mathfrak{g})$ :

$$\begin{array}{ccccc} \mathfrak{Heis}_\rho(\mathfrak{g}) & \longrightarrow & \mathfrak{Poisson}(X, \omega) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{g} & \xrightarrow{\rho} & \mathcal{X}_{\text{Ham}}(X) & \xrightarrow{\iota_{(-)}\omega} & \mathbf{BH}(X, {}^b\mathbf{B}^{n-1}\mathbb{R}) \end{array} .$$

We discuss examples of this below in 2.6.1.

## 2.6 Examples

We briefly indicate some examples of applications of the higher prequantum geometry that we have developed here.

Since in higher prequantum theory local Lagrangians are “fully de-transgressed” to higher prequantum bundles, conversely every example induces its corresponding transgressions. In the following we always start with a higher extended Chern-Simons-type theory in the sense of [13] and consider then its first transgression. As in the discussion in [14] this first transgression is the higher prequantum bundle of the topological sector of a higher extended Wess-Zumino-Witten type theory. In this way our examples appear at least in pairs as shown in the following table:

	Higher CS-type theory	higher WZW-type theory
2.6.1	3d $G$ -Chern-Simons theory	2d WZW-model on $G$
2.6.2	$\infty$ -CS theory from $L_\infty$ -integration	
2.6.3	2d Poisson Chern-Simons theory	1d quantum mechanics
2.6.4	7d String-Chern-Simons theory	6d theory related to M5-brane

### 2.6.1 Higher prequantum 2d WZW model and the smooth string 2-group

In the introduction in 1.2 we remarked that an old motivation for what we call higher prequantum geometry here is the desire to “de-transgress” the traditional construction of positive energy loop group representations of simply connected compact Lie groups  $G$  by, in our terminology, regarding the canonical  $\mathbf{BU}(1)$ -2-bundle on  $G$  (the “WZW gerbe”) as a prequantum 2-bundle. Here we discuss how prequantum 2-states for the WZW sigma-model provide at least a partial answer to this question. Then we analyze the quantomorphism 2-group of this model.

For  $G$  a connected and simply connected compact Lie group such as  $G = \text{Spin}(n)$  for  $n \geq 3$  or  $G = \text{SU}(n)$ , the first nontrivial cohomology class of the classifying space  $BG$  is in degree 4:  $H^4(BG, \mathbb{Z}) \simeq \mathbb{Z}$ . For  $\text{Spin}(n)$  the generator here is known as the *first fractional Pontryagin class*  $\frac{1}{2}p_1$ , while for  $\text{SU}(n)$  it is the second Chern class  $c_2$ . In [8] was constructed a smooth and differential lift of this class to the  $\infty$ -topos  $\text{Smooth}\infty\text{Grpd}$

from example 2.1.2 in the sense discussed in 2.3, namely a diagram of smooth higher moduli stacks of the form

$$\begin{array}{ccc}
\mathbf{BSpin}_{\text{conn}} & \xrightarrow{\frac{1}{2}\widehat{\mathbf{p}}_1} & \mathbf{B}^3U(1)_{\text{conn}} \\
\downarrow u_{\mathbf{BSpin}} & & \downarrow u_{\mathbf{B}^2U(1)} \\
\mathbf{BSpin} & \xrightarrow{\frac{1}{2}\mathbf{p}_1} & \mathbf{B}^3U(1) \\
\downarrow f & & \downarrow f \\
BSpin & \xrightarrow{\frac{1}{2}p_1} & K(\mathbb{Z}, 4)
\end{array}
\quad \begin{array}{c} \nabla_{\text{CS}} \\ \nabla_{\text{CS}}^0 \\ \int \nabla_{\text{CS}}^0 \end{array}
\quad \begin{array}{ccc}
\mathbf{BSU}_{\text{conn}} & \xrightarrow{\widehat{\mathbf{c}}_2} & \mathbf{B}^3U(1)_{\text{conn}} \\
\downarrow u_{\mathbf{BSU}} & & \downarrow u_{\mathbf{B}^2U(1)} \\
\mathbf{BSU} & \xrightarrow{\mathbf{c}_2} & \mathbf{B}^3U(1) \\
\downarrow f & & \downarrow f \\
BSU & \xrightarrow{c_2} & K(\mathbb{Z}, 4)
\end{array}$$

Here  $\int$  is the geometric realization map of example 2.3.1, and  $u_{(-)}$  is the forgetful map from the higher moduli stacks of higher principal connections to that of higher principal bundles of def. 2.3.8.

In [13, 14] we discussed that this 3-connection on the smooth moduli stack of  $G$ -principal connections – which for unspecified  $G$  we now denote by  $\nabla$  – is the full de-transgression of the (off-shell) prequantum 1-bundle of  $G$ -Chern-Simons theory, hence is the localized incarnation of 3d  $G$ -Chern-Simons theory in higher prequantum theory. In particular it is a  $\mathbf{B}^2U(1)$ -prequantization, according to def. 2.3.8 and example 2.3.10, of the Killing form invariant polynomial  $\langle -, - \rangle$  of  $G$ , which is a differential 4-form (hence a *pre-3-plectic form* in the sense of def. 2.5.7) on the moduli stack of fields:

$$\begin{array}{ccc}
& & \mathbf{B}^3U(1)_{\text{conn}} \\
& \nearrow \nabla_{\text{CS}} & \downarrow F_{(-)} \\
\mathbf{BG}_{\text{conn}} & \xrightarrow{\langle F_{(-)}, F_{(-)} \rangle} & \Omega_{\text{cl}}^4
\end{array}$$

This 3-connection on the moduli stack of  $G$ -principal connections does not descend to the moduli stack  $\mathbf{BG}$  of just  $G$ -principal bundles; it does however descend [22] as a “3-connection without top-degree forms” as in def. 2.3.13:

$$\begin{array}{ccc}
& & \mathbf{B}(\mathbf{B}^2U(1)_{\text{conn}}) \\
& \nearrow \nabla_{\text{CS}}^2 & \downarrow \mathbf{B}F_{(-)} \\
\mathbf{BG} & \longrightarrow & \mathbf{B}\Omega_{\text{cl}}^3
\end{array}$$

Therefore over the universal moduli stack of Chern-Simons fields  $\mathbf{BG}_{\text{conn}}$  we canonically have a higher quantomorphism groupoid  $\text{At}(\nabla_{\text{CS}})_\bullet$  as in 2.4.1, while over the universal moduli stack of just the “instanton sectors” of fields we have just a Courant 3-groupoid  $\text{At}(\nabla_{\text{CS}}^2)_\bullet$  as in 2.4.2. This kind of phenomenon we re-encounter below in 2.6.3.

By [13, 14], following [23], the transgression of  $\nabla_{\text{CS}}$  to maps out of the circle  $S^1$  is found to be the “WZW gerbe”, the canonical circle 2-bundle with connection  $\nabla_{\text{WZW}}$  on the Lie group  $G$  itself. We may obtain this either as the fiber integration of  $\nabla_{\text{CS}}$  restricted along the inclusion of  $G$  as the constant  $\mathfrak{g}$ -connection on the circle

$$\nabla_{\text{WZW}} : G \longrightarrow [S^1, \mathbf{BG}_{\text{conn}}] \xrightarrow{[S^1, \nabla_{\text{CS}}]} [S^1, \mathbf{B}^3U(1)_{\text{conn}}] \xrightarrow{\exp(2\pi i \int_{S^1} (-))} \mathbf{B}^2U(1)_{\text{conn}}$$

or equivalently we obtain it as the looping of  $\nabla_{\text{CS}}^2$ :

$$\nabla_{\text{WZW}} : G \simeq \Omega \mathbf{BG} \xrightarrow{\Omega \nabla_{\text{CS}}^2} \Omega \mathbf{B}(\mathbf{B}^2U(1)_{\text{conn}}) \simeq \mathbf{B}^2U(1)_{\text{conn}} .$$

This  $\nabla_{\text{WZW}}$  is the background gauge field of the 2d Wess-Zumino-Witten sigma-model, the “Kalb-Ramon B-field” under which the string propagating on  $G$  is charged. We now regard this as the  $\mathbf{BU}(1)$ -prequantization

(def. 2.3.8, example 2.3.10) of the canonical 3-form  $\langle -, [-, -] \rangle$  on  $G$  (a 2-plectic form):

$$\begin{array}{ccc} & & \mathbf{B}^2U(1)_{\text{conn}} \\ & \nearrow \nabla_{\text{WZW}} & \downarrow F_{(-)} \\ G & \xrightarrow{\langle -, [-, -] \rangle} & \Omega_{\text{cl}}^3 \end{array} .$$

By example 2.1.36 the prequantum 2-states of the prequantum 2-bundle  $\nabla_{\text{WZW}}$  are twisted unitary bundles with connection (twisted K-cocycles, after stabilization): the *Chan-Paton gauge fields*. More explicitly, with the notation as introduced there, a prequantum 2-state  $\Psi$  of the WZW model supported over a D-brane submanifold  $Q \hookrightarrow G$  is a map  $\Psi : \nabla_{\text{WZW}}|_Q \rightarrow \mathbf{d}\mathbf{d}_{\text{conn}}$  in the slice over  $\mathbf{B}^2U(1)_{\text{conn}}$ , hence a diagram of the form

$$\begin{array}{ccc} Q & \xrightarrow{\Psi} & \coprod_n (\mathbf{B}U(n) // \mathbf{B}U(1))_{\text{conn}} \\ \downarrow & \nearrow \nabla_{\text{WZW}} & \downarrow \\ G & \xrightarrow{[S^1, \mathbf{B}G_{\text{conn}}]} & \mathbf{B}^2U(1)_{\text{conn}} \\ \downarrow & \downarrow \text{conc.} & \uparrow \exp(\int_{S^1} [S^1, \nabla]) \\ G //_{\text{ad}} G & \xrightarrow{\simeq} & G\mathbf{Conn}(S)^1 \end{array} .$$

Here we have added at the bottom the map to the differential concretification of the transgressed moduli stack of fields, according to example 2.3.20. As indicated, this exhibits  $G$  as fibered over its homotopy quotient by its adjoint action. The D-brane inclusion  $Q \rightarrow G$  in the diagram is the homotopy fiber over a full point of  $G //_{\text{ad}} G$  precisely if it is a conjugacy class of  $G$ , hence a “symmetric D-brane” for the WZW model. In summary this means that this single diagram exhibiting WZW prequantum-2-states as slice maps encodes all the WZW D-brane data as discussed in the literature [24]. In particular, in [14] we showed that the transgression of these prequantum 2-states  $\Psi$  to prequantum 1-states over the loop group  $LG$  naturally encodes the anomaly cancellation of the open bosonic string in the presence of D-branes (the Kapustin-part of the Freed-Witten-Kapustin quantum anomaly cancellation).

Notice that in [25] it is shown that the ring of positive energy representations of the loop group of  $G$  is generated by the push-forward in K-theory of these twisted bundles over conjugacy classes of  $G$ . Taken together this provides at least some aspects of an answer to the question in 1.2, concerning a higher stacky refinement of the geometric construction of loop group representation theory.

We may now study the quantomorphism 2-group of  $\nabla_{\text{WZW}}$ , def. 2.4.3, on these 2-states, hence, in the language of twisted cohomology, the 2-group of twist automorphism. First, one sees that by inspection that this is the action that integrates and globalizes the D-brane gauge transformations which are familiar from the string theory literature, where the local connection 1-form  $A$  on the twisted bundle is shifted and the local connection 2-form on the prequantum bundle transforms as

$$A \mapsto A + \lambda, \quad B \mapsto B + d\lambda.$$

In order to analyze the quantomorphism 2-group here in more detail, notice that since the 2-plectic form  $\langle -, [-, -] \rangle \in \Omega_{\text{cl}}^3(G)$  is a left invariant form (by definition), the left action of  $G$  on itself is Hamiltonian, in the sense of def. 2.4.5, and so we have the corresponding Heisenberg 2-group  $\mathbf{Heis}(G, \nabla_{\text{WZW}})$  of def. 2.4.5 inside the quantomorphism 2-group. By theorem 2.5.1 this is a 2-group extension of  $G$  of the form

$$U(1)\mathbf{FlatConn}(G) \longrightarrow \mathbf{Heis}(\nabla_{\text{WZW}}) \longrightarrow G .$$

Since  $G$  is connected and simply connected, there is by prop. 2.5.4 an equivalence of smooth 2-groups  $U(1)\mathbf{FlatConn}(G) \simeq \mathbf{B}U(1)$  and so the WZW Heisenberg 2-group is in fact a smooth circle 2-group extension

$$\mathbf{B}U(1) \longrightarrow \mathbf{Heis}(\nabla_{\text{WZW}}) \longrightarrow G$$

classified by a cocycle  $\mathbf{B}(\nabla_{\text{WZW}} \circ (-)) : \mathbf{B}G \rightarrow \mathbf{B}^3U(1)$ . If  $G$  is compact and simply connected, then by [D, section 4.4.6.2]  $\pi_0 \mathbf{H}(\mathbf{B}G, \mathbf{B}^3U(1)) \simeq H^4(BG, \mathbb{Z}) \simeq \mathbb{Z}$ . This integer is the *level*, the cocycle corresponding to the generator  $\pm 1$  is  $\frac{1}{2}\mathbf{p}_1$  for  $G = \text{Spin}$  and  $\mathbf{c}_2$  for  $G = SU$ . The corresponding extension is the String 2-group extension of example 2.1.17

$$\mathbf{B}U(1) \longrightarrow \text{String}_G \longrightarrow G .$$

Accordingly, under Lie differentiation, one finds (this was originally observed in [42], a re-derivation in the context of def. 2.5.21 is in [L]) that the Heisenberg Lie 2-algebra extension of theorem 2.5.18 combined with def. 2.5.21 is the *string Lie 2-algebra* extension (see example 2.1.17)

$$\begin{array}{c} \mathbf{B}\mathbb{R} \longrightarrow \mathfrak{Heis}_{\langle -, [-, -] \rangle}(\mathfrak{g}) \longrightarrow \mathfrak{g} . \\ \simeq \\ \text{string}_{\mathfrak{g}} \end{array}$$

## 2.6.2 Higher prequantum and Chern-Simons-type theories and $L_\infty$ -algebra cohomology

The construction of the higher prequantum bundle  $\nabla_{\text{CS}}$  for Chern-Simons field theory in 2.6.1 above follows a general procedure – which might be called *differential Lie integration of  $L_\infty$ -cocycles* – that produces a whole class of examples of natural higher prequantum geometries: namely those *extended higher Chern-Simons-type field theories* which are encoded by an  $L_\infty$ -invariant polynomial on an  $L_\infty$ -algebra, in generalization of how ordinary  $G$ -Chern-Simons theory for a simply connected simple Lie group  $G$  is all encoded by the Killing form invariant polynomial (and as opposed to for instance to the cup product higher  $U(1)$ -Chern-Simons theories, see [13]). Since also the following two examples in 2.6.3 and 2.6.4 are naturally obtained this way, we here briefly recall this construction, due to [8], with an eye towards its interpretation in higher prequantum geometry.

Given an  $L_\infty$ -algebra  $\mathfrak{g} \in L_\infty$ , there is a natural notion of sheaves of (flat)  $\mathfrak{g}$ -valued smooth differential forms

$$\Omega_{\text{flat}}(-, \mathfrak{g}) \hookrightarrow \Omega(-, \mathfrak{g}) \in \text{Sh}(\text{SmthMfd}) ,$$

and this is functorial in  $\mathfrak{g}$  (for the correct (“weak”) homomorphisms of  $L_\infty$ -algebras). Therefore there is a functor – denoted  $\exp(-)$  in [8] – which assigns to an  $L_\infty$ -algebra  $\mathfrak{g}$  the presheaf of Kan complexes that over a test manifold  $U$  has as set of  $k$ -cells the set of those smoothly  $U$ -parameterized collections of flat  $\mathfrak{g}$ -valued differential forms on the  $k$ -simplex  $\Delta^k$  which are sufficiently well behaved towards the boundary of the simplex (have “sitting instants”). Under the presentation  $L_{\text{lie}}[\text{SmoothMfd}^{\text{op}}] \simeq \text{Smooth}\infty\text{Grpd}$  of the  $\infty$ -topos of smooth  $\infty$ -groupoids in example 2.1.2 this yields a Lie integration construction from  $L_\infty$ -algebras to smooth  $\infty$ -groupoids. (So far this is the fairly immediate stacky and smooth refinement of a standard construction in rational homotopy theory and deformation theory, see the references in [8] for a list of predecessors of this construction.)

In higher analogy to ordinary Lie integration, one finds that  $\exp(\mathfrak{g})$  is the “geometrically  $\infty$ -connected” Lie integration of  $\mathfrak{g}$ : the geometric realization  $\int \exp(\mathfrak{g})$ , example 2.3.1, of  $\exp(\mathfrak{g}) \in L_{\text{lie}}[\text{SmoothMfd}^{\text{op}}, \text{KanCplx}] \simeq \text{Smooth}\infty\text{Grpd}$  is always contractible. For instance for  $\mathfrak{g} = \mathbb{R}[-n+1] = \mathbf{B}^{n-1}\mathbb{R}$  the abelian  $L_\infty$ -algebra concentrated on  $\mathbb{R}$  in the  $n$ th degree, we have

$$\exp(\mathbb{R}[-n+1]) \simeq \mathbf{B}^n\mathbb{R} \in \text{Smooth}\infty\text{Grpd}$$

and by example 2.3.1 it follows that  $\int \mathbf{B}^n\mathbb{R} \simeq B^n\mathbb{R} \simeq *$ . Geometrically non- $\infty$ -connected Lie integrations of  $\mathfrak{g}$  arise notably as truncations of the  $\infty$ -stack  $\exp(\mathfrak{g})$ , according to remark 2.1.5. For instance for  $\mathfrak{g}_1$  an ordinary Lie algebra, then the 1-truncation of the  $\infty$ -stack  $\exp(\mathfrak{g}_1)$  to a stack of 1-groupoids reproduces (the internal delooping of) the simply connected Lie group  $G$  corresponding to  $\mathfrak{g}$  by ordinary Lie theory:

$$\tau_1 \exp(\mathfrak{g}_1) \simeq \mathbf{B}G \in \text{Smooth}\infty\text{Grpd} .$$

Similarly for  $\mathbf{string} \in L_\infty \mathbf{Alg}$  the string Lie 2-algebra, example 2.1.17, the 2-truncation of its universal Lie integration to a stack of 2-groupoids reproduces the moduli stack of String-principal 2-bundles:

$$\tau_2 \exp(\mathbf{string}) \simeq \mathbf{BString} \in \mathbf{Smooth}\infty\mathbf{Grpd}.$$

Now the simple observation that yields the analogous Lie integration of  $L_\infty$ -cocycles is that a degree- $n$   $L_\infty$ -cocycle  $\mu$  on an  $L_\infty$ -algebra  $\mathfrak{g}$  is equivalently a map of  $L_\infty$ -algebras of the form

$$\mu : \mathbf{Bg} \rightarrow \mathbf{B}^n \mathbb{R};$$

and since  $\exp(-)$  is a functor, every such cocycle immediately integrates to a morphism

$$\exp(\mu) : \exp(\mathfrak{g}) \rightarrow \mathbf{B}^n \mathbb{R}$$

in  $\mathbf{Smooth}\infty\mathbf{Grpd}$ , hence by remark 2.1.4 to a universal cocycle on the smooth moduli  $\infty$ -stack  $\exp(\mathfrak{g})$ . Moreover, this cocycle descends to the  $n$ -truncation of its domain as a  $\mathbb{R}/\Gamma$  cocycle on the resulting moduli  $n$ -stack

$$\exp(\mu) : \tau_n \exp(\mathfrak{g}) \rightarrow \mathbf{B}^n(\mathbb{R}/\Gamma),$$

where  $\Gamma \hookrightarrow \mathbb{R}$  is the period lattice of the cocycle  $\mu$ .

For instance for

$$\langle -, [-, -] \rangle : \mathbf{Bg}_1 \rightarrow \mathbf{B}^3 \mathbb{R}$$

the canonical 3-cocycle on a semisimple Lie algebra (where  $\langle -, - \rangle$  is the Killing form invariant polynomial as before), its period group is  $\pi_3(G) \simeq \mathbb{Z}$  of the simply connected Lie group  $G$  integrating  $\mathfrak{g}_1$ , and hence the Lie integration of the 3-cocycle yields a map of smooth  $\infty$ -stacks of the form

$$\exp(\langle -, [-, -] \rangle) : \mathbf{BG} \xrightarrow{\simeq} \tau_3 \exp(\mathfrak{g}_1) \quad \mathbf{B}^3(\mathbb{R}/\mathbb{Z}) = \mathbf{B}^3 U(1),$$

where we use that for the connected and simply connected Lie group  $G$  not only the 1-truncation but also still the 3-truncation of  $\exp(\mathfrak{g}_1)$  gives the delooping stack:  $\tau_3 \exp(\mathfrak{g}_1) \simeq \tau_2 \exp(\mathfrak{g}_1) \simeq \tau_1 \exp(\mathfrak{g}_1) \simeq \mathbf{BG}$ .

Indeed, this is what yields the refinement of the generator  $c : \mathbf{BG} \rightarrow K(\mathbb{Z}, 4)$  to smooth cohomology, which we used above in 2.6.1, for instance for  $\mathfrak{g}_1 = \mathfrak{so}$  the Lie algebra of the Spin group, the Lie integration of its canonical Lie 3-cocycle

$$\exp(\langle -, [-, -] \rangle_{\mathfrak{so}}) \simeq \frac{1}{2} \mathbf{p}_1$$

yields the smooth refinement of the first fractional universal Pontryagin class.

This is shown in [8] by further refining the  $\exp(-)$ -construction to one that yields not just moduli  $\infty$ -stacks of  $G$ -principal  $\infty$ -bundles, but yields their differential refinements. The key to this construction is the observation that an invariant polynomial  $\langle -, \dots, - \rangle$  on a Lie algebra and more generally on an  $L_\infty$ -algebra  $\mathfrak{g}$  yields a *globally* defined (hence invariant) differential form on the moduli  $\infty$ -stack  $\mathbf{BG}_{\text{conn}}$ :

$$\langle F_{(-)}, \dots, F_{(-)} \rangle : \mathbf{BG}_{\text{conn}} \rightarrow \Omega_{\text{cl}}^{n+1}.$$

In components this is simply given, as the notation is supposed to indicate, by sending a  $G$ -principal connection  $\nabla$  first to its  $\mathfrak{g}$ -valued curvature form  $F_\nabla$  and then evaluating that in the invariant polynomial. In fact this property is part of the *definition* of  $\mathbf{BG}_{\text{conn}}$  for the non-braided  $\infty$ -groups  $G$ . This we think of as a higher analog of Chern-Weil theory in higher differential geometry. We may also usefully think of the invariant polynomial  $\langle F_{(-)}, \dots, F_{(-)} \rangle$  as being a pre  $n$ -plectic form on the moduli stack  $\mathbf{BG}_{\text{conn}}$ , in evident generalization of the terminology for smooth manifolds in def. 2.5.7.

Using this, there is a differential refinement  $\exp(-)_{\text{conn}}$  of the  $\exp(-)$ -construction, which lifts this pre- $n$ -plectic form to differential cohomology and hence provides its pre-quantization, according to def. 2.3.8:

$$\begin{array}{ccc} & \mathbf{B}^n(\mathbb{R}/\Gamma)_{\text{conn}} & \\ \exp(\mu)_{\text{conn}} \nearrow & \downarrow F_{(-)} & \\ \tau_n \exp(\mathfrak{g})_{\text{conn}} & \xrightarrow{\langle F_{(-)} \wedge \dots \wedge F_{(-)} \rangle} & \Omega_{\text{cl}}^{n+1} \end{array}.$$



Here the higher stack  $\exp(-)_{\text{conn}}$  assigns to a test manifold  $U$  smoothly  $U$ -parameterized collections of simplicial  $L_\infty$ -Ehresmann connections: the  $k$ -cells of  $\exp(\mathfrak{g})_{\text{conn}}$  are  $\mathfrak{g}$ -valued differential forms  $A$  on  $U \times \Delta^k$  (now not necessarily flat) satisfying an  $L_\infty$ -analog of the two conditions on a traditional Ehresmann connection 1-form: restricted to the fiber (hence the simplex) the  $L_\infty$ -form datum becomes flat, and moreover the curvature invariants  $\langle F_A \wedge \dots \wedge F_A \rangle$  obtained by evaluating the  $L_\infty$ -curvature forms in all  $L_\infty$ -invariant polynomials descends down the simplex bundle  $U \times \Delta^k \rightarrow U$ .

For example the differential refinement of the prequantum 3-bundle of 3d  $G$ -Chern-Simons theory  $\frac{1}{2}\mathbf{p}_1 \simeq \tau_3 \exp(\langle -, [-, -] \rangle)$  obtained this way is the universal Chern-Simons 3-connection

$$\exp(\langle -, [-, -] \rangle_{\mathfrak{so}})_{\text{conn}} \simeq \frac{1}{2}\hat{\mathbf{p}}_1 : \mathbf{B}\text{Spin}_{\text{conn}} \rightarrow \mathbf{B}^3U(1)_{\text{conn}}$$

whose transgression to codimension 0 is the standard Chern-Simons action functional, as discussed above in 2.6.1. Analogously, the differential Lie integration of the next cocycle, the canonical 7-cocycle, but now regarded as a cocycle on  $\mathbf{string}$ , yields a prequantum 7-bundle on the moduli stack of String-principal 2-connections:

$$\exp(\langle -, [-, -], [-, -], [-, -] \rangle_{\mathfrak{so}})_{\text{conn}} \simeq \frac{1}{6}\hat{\mathbf{p}}_2 : \mathbf{B}\text{String}_{\text{conn}} \rightarrow \mathbf{B}^7U(1)_{\text{conn}}.$$

This defines a 7-dimensional nonabelian Chern-Simons theory, which we come to below in 2.6.4.

In conclusion this means that  $L_\infty$ -algebra cohomology is a direct source of higher smooth  $(\mathbf{B}^{n-1}(\mathbb{R}/\Gamma))$ -prequantum geometries on higher differential moduli stacks. For  $\mu$  any degree- $n$   $L_\infty$ -cocycle on an  $L_\infty$ -algebra  $\mathfrak{g}$ , differential Lie integration yields the higher prequantum bundle

$$\exp(\mu)_{\text{conn}} : \tau_n \exp(\mathfrak{g})_{\text{conn}} \rightarrow \mathbf{B}^n(\mathbb{R}/\Gamma).$$

Moreover, these are by construction higher prequantum bundles for higher Chern-Simons-type higher gauge theories in that their transgression to codimension 0

$$\exp\left(\int_{\Sigma_n} [\Sigma_n, \exp(\mu)_{\text{conn}}]\right) : [\Sigma_n, \tau_n \exp(\mathfrak{g})_{\text{conn}}] \longrightarrow \mathbb{R}/\Gamma$$

is an action functional on the stack of  $\mathfrak{g}$ -gauge fields  $A$  on a given closed oriented manifold  $\Sigma_n$  which is locally given by the integral of a Chern-Simons  $(n-1)$ -form  $\text{CS}_\mu(A)$  (with respect to the corresponding  $L_\infty$ -invariant polynomial) and globally given by a higher-gauge consistent globalization of such integrals.

All of this discussion generalizes verbatim from  $L_\infty$ -algebras to  $L_\infty$ -algebroids, too. In [9] it was observed that therefore all the perturbative field theories known as *AKSZ sigma-models* have a Lie integration to what here we call higher prequantum bundles for higher Chern-Simons type field theories: these are precisely the cases as above where  $\mu$  transgresses to a binary invariant polynomial  $\langle -, - \rangle$  on the  $L_\infty$ -algebroid which non-degenerate. On the level of globally defined differential forms this is [43]. In the next section 2.6.3 we consider one low-dimensional example in this family and observe that its higher geometric prequantum and quantum theory has secretly been studied in some detail already – but in 1-geometric disguise.

For higher Chern-Simons action functionals  $\exp(\mu)_{\text{conn}}$  as above, one finds that their variational differential at a field configuration  $A$  given by globally defined differential form data is proportional to

$$\delta \exp\left(\int_{\Sigma_{n-1}} [\Sigma_n, \exp(\mu)_{\text{conn}}]\right) \propto \int_{\Sigma} \langle F_A \wedge \dots \wedge F_A \wedge \delta A \rangle.$$

Therefore the Euler-Lagrange equations of motion of the corresponding  $n$ -dimensional Chern-Simons theory assert that

$$\langle F_A \wedge \dots \wedge F_A, - \rangle = 0.$$

(Notice that in general  $F_A$  is an inhomogenous differential form, so that this equation in general consists of several independent components.) In particular, if the invariant polynomial is binary, hence of the form  $\langle -, - \rangle$ , and furthermore non-degenerate (this is precisely the case in which the general  $\infty$ -Chern-Simons theory reproduces the AKSZ  $\sigma$ -models), then the above equations of motion reduce to

$$F_A = 0$$

and hence assert that the critical/on-shell field configurations are precisely those  $L_\infty$ -algebroid valued connections which are flat.

In this case the higher moduli stack  $\tau_n \exp(\mathfrak{g})$ , which in general is the moduli stack of instanton/charge-sectors underlying the topologically nontrivial  $\mathfrak{g}$ -connections, acquires also a different interpretation. By the above discussion, its  $(n-1)$ -cells are equivalently flat  $\mathfrak{g}$ -valued connections on the  $(n-1)$ -disks and its  $n$ -cells implement gauge equivalences between such data. But since the equations of motion  $F_A = 0$  are first order differential equations, flat connections on  $D^{n-1}$  bijectively correspond to critical field configuration on the cylinder  $D^{n-1} \times [-T, T]$ . Therefore the collection of  $(n-1)$ -cells of  $\tau_n \exp(\mathfrak{g})$  is the higher/extended *covariant phase space* for “open genus-0  $(n-1)$ -branes” in the model. Moreover, the  $n$ -cells between these  $(n-1)$ -cells implement the gauge transformations on such initial value data and hence  $\tau_n \exp(\mathfrak{g})$  is, in codimension 1, the higher/extended *reduced phase space* of the model in codimension 1. For  $n = 2$  this perspective was amplified in [44], we turn to this special case below in 2.6.3).

As an example, from this perspective the construction of the WZW-gerbe by looping as discussed above in 2.6.2 is equivalently the construction of the on-shell prequantum 2-bundle in codimension 2 for “Dirichlet boundary conditions” for the open Chern-Simons membrane. Namely  $\mathbf{B}G$  is now the extended reduced phase space, and so the extended phase space of membranes stretching between the unique point is the homotopy fiber product of the two point inclusions  $Q_0 \longrightarrow \mathbf{B}G \longleftarrow Q_1$ , with  $Q_0, Q_1 = *$ , hence is  $\Omega \mathbf{B}G \simeq G$ . Since the on-shell prequantum 2-bundle  $\nabla_{\text{CS}}^1$  trivializes over these inclusions, as exhibited by diagrams

$$\begin{array}{ccc} Q_i & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathbf{B}G & \xrightarrow{\nabla_{\text{CS}}^2} & \mathbf{B}^3 U(1)_{\text{conn}^2} \end{array},$$

the on-shell prequantum 3-bundle  $\nabla_{\text{CS}}^2$  extends to a diagram of relative cocycles of the form

$$\begin{array}{ccc} Q_0 & \longrightarrow & * \\ \downarrow & & \downarrow \\ \mathbf{B}G & \xrightarrow{\nabla_{\text{CS}}^2} & \mathbf{B}(\mathbf{B}^3 U(1)_{\text{conn}}) \\ \uparrow & & \uparrow \\ Q_1 & \longrightarrow & * \end{array},$$

hence, under forming homotopy fiber products, to the WZW-2-connection  $\Omega \nabla_{\text{CS}}^2 : G \rightarrow \mathbf{B}^2 U(1)$  on the extended phase space  $G$ .

In the next section we see another example of this phenomenon.

### 2.6.3 Higher prequantum 2d Poisson Chern-Simons theory and quantum mechanics

We discuss here how the higher geometric quantization of a stacky refinement of the 2d Poisson sigma-model QFT yields, holographically, the strict deformation quantization of the underlying Poisson manifold, hence of a 1-dimensional field theory (quantum mechanics). We do so by unwinding what higher prequantum theory says in this case, expressed in components in ordinary prequantum theory, and observing then that in terms of this disguised form the higher prequantization and its holographic relation to 1d quantization has already been worked out, secretly, in [46]. (At least roughly this relation had previously been voiced in the introduction of [44], but at that time the geometric quantization of symplectic groupoids as in [46] had yet to be fully understood.)

A non-degenerate and binary invariant polynomial which induces a pre-2-plectic structure on the moduli stack of a higher Chern-Simons type theory

$$\omega := \langle F(-)F(-) \rangle : \tau_2 \exp(\mathfrak{P})_{\text{conn}} \rightarrow \Omega_{\text{cl}}^3$$

exists precisely on Poisson Lie algebroids  $\mathfrak{P}$ , induced from Poisson manifolds  $(X, \pi)$ . The differential Lie integration method described above yields a  $(\mathbf{B}(\mathbb{R}/\Gamma))$ -prequantization

$$\begin{array}{ccc} & \mathbf{B}^2(\mathbb{R}/\Gamma)_{\text{conn}} & \\ \nearrow \nabla_P & \downarrow F(-) & \\ \tau_1 \exp(\mathfrak{P})_{\text{conn}} & \xrightarrow{\omega} & \Omega_{\text{cl}}^3 \end{array} \quad .$$

The action functional of this higher prequantum field theory over a closed oriented 2-dimensional smooth manifold  $\Sigma_2$  is, again by [13, 9], the transgression of the higher prequantum bundle to codimension 0

$$\exp \left( \int_{\Sigma_2} [\Sigma_2, \nabla_P] \right) : [\Sigma_2, \tau_1 \exp(\mathfrak{P})_{\text{conn}}] \longrightarrow \mathbb{R}/\Gamma \quad .$$

We observe now that two complementary sectors of this higher prequantum 2d Poisson Chern-Simons field theory  $\nabla_P$  lead a separate life of their own in the literature: on the one hand the sector where the bundle structures and hence the nontrivial “instanton sectors” of the field configurations are ignored and only the globally defined connection differential form data is retained; and on the other hand the complementary sector where only these bundle structures/ instanton sectors are considered and the connection data is ignored:

1. The restriction of the action functional  $\exp(\int_{\Sigma_2} [\Sigma_2, \nabla_P])$  to the linearized theory – hence along the canonical inclusion  $\Omega(\Sigma, \mathfrak{P}) \hookrightarrow [\Sigma_2, \exp(\mathfrak{P})_{\text{conn}}]$  of globally defined  $\mathfrak{P}$ -valued forms into all  $\exp(\mathfrak{P})$ -principal connections – is the action functional of the *Poisson sigma-model* [43].
2. The restriction of the moduli stack of fields  $\tau_1 \exp(\mathfrak{P})_{\text{conn}}$  to just  $\tau_1 \exp(\mathfrak{P})$  obtained by forgetting the differential refinement (the connection data) and just remembering the underlying  $\exp(\mathfrak{P})$ -principal bundles, yields what is known as the *symplectic groupoid* of  $\mathfrak{P}$ .

Precisely: while the prequantum 2-bundle  $\nabla_P$  does not descend along the forgetful map  $\tau_1 \exp(\mathfrak{P})_{\text{conn}} \rightarrow \tau_1 \exp(\mathfrak{P})$  from moduli of  $\tau_1 \exp(\mathfrak{P})$ -principal connections to their underlying  $\tau_1(\exp(\mathfrak{P}))$ -principal bundles, its version  $\nabla_P^1$  “without curving”, given by def. 2.3.13, does descend (this is as for 3d Chern-Simons theory discussed above in 2.6.1) and so does hence its curvature  $\omega^1$ , which by remark 2.3.11 has coefficients in  $\mathbf{B}\Omega_{\text{cl}}^2$  instead of  $\Omega_{\text{cl}}^3$ :

$$\begin{array}{ccc} & \mathbf{B}(\mathbf{B}(\mathbb{R}/\Gamma)_{\text{conn}}) & \\ \nearrow \nabla_P^1 & \downarrow \mathbf{B}F(-) & \\ \tau_1 \exp(\mathfrak{P}) & \xrightarrow{\omega^1} & \mathbf{B}\Omega_{\text{cl}}^2 \end{array} \quad .$$

If here the smooth groupoid  $\tau_1 \exp(\mathfrak{P}) \in \text{Smooth}\infty\text{Grpd}$  happens to have a presentation by a Lie groupoid under the canonical inclusion of Lie groupoids into smooth  $\infty$ -groupoids of example 2.1.2 (this is an integrability condition on  $\mathfrak{P}$ ) then equipped with the de Rham hypercohomology 3-cocycle  $\omega^1$  it is called in the literature a *pre-quasi-symplectic groupoid* [45]. If moreover the de Rham hypercohomology 3-cocycle  $\omega^1$  – which in general is given by 3-form data and 2-form data on a Čech simplicial presheaf that resolves  $\tau_1 \exp(\mathfrak{P})$  (in generalization of the simple example 2.1.35 above) – happens to be represented by just a globally defined 2-form on the manifold of morphisms of the Lie groupoid (which is then necessarily closed and “multiplicative”), then this local data is called a (pre-)*symplectic groupoid*, see [46] for a review and further pointers to the literature.

So in the case that the descended (pre-)2-plectic form  $\omega^1 : \tau_1 \exp(\mathfrak{P}) \rightarrow \mathbf{B}\Omega_{\text{cl}}^2$  of the higher prequantum 2d Poisson Chern-Simons theory is represented by a multiplicative symplectic 2-form on the manifold of morphisms of the Lie groupoid  $\tau_1 \exp(\mathfrak{P})$ , then one is faced with a situation that looks like ordinary symplectic geometry subject to a kind of equivariance condition. This is the perspective from which symplectic groupoids were originally introduced and from which they are mostly studied in the literature (with the exception at least of [45], where the higher geometric nature of the setup is made explicit): as a means to re-code Poisson geometry in terms of ordinary symplectic geometry. The goal of finding a sensible geometric quantization of symplectic groupoids (and hence in some sense of Poisson manifolds, this we come back to below) was finally achieved in [46].

In order to further understand the conceptual role of the prequantum 2-bundle  $\nabla_{\mathfrak{P}}^1$ , notice that by the discussion in 2.6.2, following [44], we may think of the symplectic groupoid  $\tau_1 \exp(\mathfrak{P})$  as the extended reduced phase space of the open string Poisson-Chern-Simons theory. More precisely, if  $\mathfrak{C}_0, \mathfrak{C}_1 \hookrightarrow \mathfrak{P}$  are two sub-Lie algebroids, then the homotopy fiber product  $\mathbf{Phase}_{\mathfrak{C}_0, \mathfrak{C}_1}$  in

$$\begin{array}{ccc} & \mathbf{Phase}_{\mathfrak{C}_0, \mathfrak{C}_1} & \\ \swarrow & & \searrow \\ \tau_1 \exp(\mathfrak{C}_0) & & \tau_1 \exp(\mathfrak{C}_1) \\ \searrow & & \swarrow \\ & \tau_1 \exp(\mathfrak{P}) & \end{array}$$

should be the ordinary reduced phase space of open strings that stretch between  $\mathfrak{C}_0$  and  $\mathfrak{C}_1$ , regarded as D-branes. Unwinding the definitions shows that this is precisely what is shown in [47]: for  $\mathfrak{C}_0, \mathfrak{C}_1 \hookrightarrow \mathfrak{P}$  two Lagrangian sub-Lie algebroids (hence over coisotropic submanifolds of  $X$ ) the homotopy fiber product stack  $\mathbf{Phase}_{\mathfrak{C}_0, \mathfrak{C}_1}$  is the symplectic reduction of the open  $\mathfrak{C}_0$ - $\mathfrak{C}_1$ -string phase space.

Notice that the condition that  $\mathfrak{C}_i \hookrightarrow \mathfrak{P}$  be Lagrangian sub-Lie algebroids means that restricted to them the prequantum 2-bundle becomes flat, hence that we have commuting squares

$$\begin{array}{ccc} \tau_1 \exp(\mathfrak{C}_i) & \longrightarrow & \mathbf{bB}^2(\mathbb{R}/\Gamma) \\ \downarrow & & \downarrow \\ \tau_1 \exp(\mathfrak{P}) & \xrightarrow{\nabla_P^1} & \mathbf{B}(\mathbf{B}(\mathbb{R}/\Gamma)_{\text{conn}}) \end{array} .$$

If the inclusions are even such  $\nabla_P^1$  entirely trivializes over them, hence that we have diagrams

$$\begin{array}{ccc} \tau_1 \exp(\mathfrak{C}_i) & \xrightarrow{\nabla_{\mathfrak{C}_i}} & * \\ \downarrow & & \downarrow \\ \tau_1 \exp(\mathfrak{P}) & \xrightarrow{\nabla_P^1} & \mathbf{B}(\mathbf{B}(\mathbb{R}/\Gamma)_{\text{conn}}) \end{array} ,$$

then under forming homotopy fiber products the prequantum 2-bundle  $\nabla_P^1$  induces a prequantum 1-bundle on the open string phase space by the D-brane-relative looping of the on-shell prequantum 2-bundle:

$$\nabla_{\mathfrak{C}_0} \times_{\nabla_P^1} \nabla_{\mathfrak{C}_1} : \mathbf{Phase}_{\mathfrak{C}_0, \mathfrak{C}_1} \longrightarrow \mathbf{B}(\mathbb{R}/\Gamma)_{\text{conn}} .$$

We now review the steps in the geometric quantization of the symplectic groupoid due to [46] – hence the full geometric quantization of the prequantization  $\nabla_P^1$  – while discussing along the way the natural re-interpretation of the steps involved from the point of view of the higher geometric prequantum theory of 2d Poisson Chern-Simons theory.

Consider therefore  $\nabla_P^1$ , as above, as the  $(\mathbf{BU}(1))$ -prequantum 2-bundle of 2d Poisson Chern-Simons theory according to def. 2.3.8 and example 2.3.10. If we have a genuine symplectic groupoid instead of a pre-quasi-symplectic groupoid then it makes sense ask for this prequantization to be presented by a Čech-Deligne 3-cocycle on  $\tau_1 \exp(\mathfrak{P})$  which is given just by a multiplicative circle-bundle with connection on the space of morphisms of the symplectic groupoid, and otherwise trivial local data on the space of objects. While this is unlikely to be the most general higher prequantization of the 2d Poisson Chern-Simons theory, this is the choice that admits to think of the situation as if it were a setup in traditional symplectic geometry equipped with an equivariance- or “multiplicativity”-constraint, as opposed to a setup in higher 2-plectic geometry. (Such a “multiplicative circle bundle” on the space of morphisms of a Lie groupoid is just like the transition bundle that appears in the definition of a bundle gerbe, only that here the underlying groupoid is not a Čech groupoid resolving a plain manifold, but is, in general, a genuine non-trivial Lie groupoid.)

Such a multiplicative prequantum bundle is the traditional notion of prequantization of a symplectic groupoid and is also considered in [46]. The central construction there is that of the convolution  $C^*$ -algebra  $\mathcal{A}(\nabla^1)_{\text{pq}}$  of sections of the multiplicative prequantum bundle on the space of morphisms of the symplectic groupoid, and its subalgebra

$$\mathcal{A}(\nabla_P^1)_q \hookrightarrow \mathcal{A}(\nabla_P^1)_{\text{pq}}$$

of polarized sections, once a suitable kind of polarization has been chosen. Observe then that convolution algebras of sections of transition bundles of bundle gerbes have a natural interpretation in the higher geometry of the corresponding higher prequantum bundle  $\nabla^1$ : by [34, section 5] one finds that these are the algebras whose modules are the unitary bundles which are twisted by  $\nabla^1$ : the “bundle gerbe modules”.

But by remark 2.1.11 and the discussion above in 2.6.1,  $\nabla_P^1$ -twisted unitary bundles are equivalently the (pre-)quantum 2-states of  $\nabla_P^1$  regarded as a prequantum 2-bundle. These hence form a category  $\mathcal{A}(\nabla_P^1)_q \text{Mod}$  of modules, and such categories of modules are naturally interpreted as *2-modules* with *2-basis* the linear category  $\mathbf{BA}(\nabla_P^1)_q$  [48, appendix]:

$$\left\{ \begin{array}{c} \text{quantum 2-states of} \\ \text{higher prequantum 2d Poisson Chern-Simons theory} \end{array} \right\} \simeq \mathcal{A}(\nabla_P^1)_q \text{Mod} \in 2\text{Mod}.$$

This resolves what might be a conceptual puzzlement concerning the construction in [46] in view of the usual story of geometric quantization: ordinarily geometric quantization directly produces the space of states of a theory, while it requires more work to obtain the algebra of quantum observables acting on that. In [46] it superficially seems to be the other way around, an algebra drops out as a direct result of the quantization procedure. However, from the point of view of higher prequantum geometry this algebra *is* (a 2-basis for) the 2-space of 2-states; and indeed obtaining the *2-algebra* or *higher quantum operators* which would act on these 2-states does require more work (and has not been discussed yet).

Of course [46] amplifies a different perspective on the central result obtained there: that  $\mathcal{A}(\nabla_P^1)_q$  is also a *strict  $C^*$ -deformation quantization* of the Poisson manifold that corresponds to the Poisson Lie algebroid  $\mathfrak{P}!$  From the point of view of higher prequantum theory this says that the higher-geometric quantized 2d Poisson Chern-Simons theory has a 2-space of quantum 2-states in codimension 2 that encodes the correlators (commutators) of a 1-dimensional quantum mechanical system. In other words, we see that the construction in [46] is implicitly a “holographic” (strict deformation-)quantization of a Poisson manifold by directly higher-geometric quantizing instead a 2-dimensional QFT.

Notice that this statement is an analogue in  $C^*$ -deformation quantization to the seminal result on *formal* deformation quantization of Poisson manifolds: The general formula that Kontsevich had given for the formal deformation quantization of a Poisson manifold was found by Cattaneo-Felder to be the point-particle limit of the 3-point function of the corresponding 2d Poisson sigma-model [49]. A similar result is discussed in [50]. There the 2d A-model (which is a special case of the Poisson sigma-model) is shown to holographically encode the quantization of its target space symplectic manifold regarded as a 1d quantum field theory.

In summary, the following table indicates how the “holographic” formal deformation quantization of Poisson manifolds by Kontsevich-Cattaneo-Felder is analogous to the “holographic” strict deformation quantization of Poisson manifolds by [46], when reinterpreted in higher prequantum theory as discussed above.

	perturbative formal algebraic quantization	non-perturbative geometric quantization
quantization of Poisson manifold	formal deformation quantization	strict $C^*$ -deformation quantization
“holographically” related 2d field theory	Poisson sigma-model	2d Poisson Chern-Simons theory
moduli stack of fields of the 2d field theory	Poisson Lie algebroid	symplectic groupoid
quantization of holographically related 2d field theory	perturbative quantization of Poisson sigma-model	higher geometric quantization of 2d Poisson Chern-Simons theory
1d observable algebra is holographically identified with...	point-particle limit of 3-point function	basis for 2-space of quantum 2-states

More details on this higher geometric interpretation of traditional symplectic groupoid quantization will appear in [B, N].

#### 2.6.4 Higher prequantum 6d WZW-type models and the smooth fivebrane-6-group

We close the overview of examples by providing a brief outlook on higher dimensional examples in general, and on certain higher prequantum field theories in dimensions seven and six in particular.

To appreciate the following pattern, recall that in 2.6.1 above we discussed how the universal  $G$ -Chern-Simons ( $\mathbf{B}^2U(1)$ )-principal connection  $\nabla_{\text{CS}}$  over  $\mathbf{B}G_{\text{conn}}$  transgresses to the Wess-Zumino-Witten  $\mathbf{B}U(1)$ -principal connection  $\nabla_{\text{WZW}}$  on  $G$  itself. At the level of the underlying principal  $\infty$ -bundles  $\nabla_{\text{CS}}^0$  and  $\nabla_{\text{WZW}}^0$  this relation holds very generally:

for  $G \in \text{Grp}(\mathbf{H})$  any  $\infty$ -group, and  $A \in \text{Grp}_{n+1}(\mathbf{H})$  any sufficiently highly deloopable  $\infty$ -group (def. 2.1.18) in any  $\infty$ -topos  $\mathbf{H}$ , consider a class in smooth  $\infty$ -group cohomology (remark 2.1.20)

$$c \in H_{\text{grp}}^{n+1}(G, A) = H^{n+1}(\mathbf{B}G, A),$$

hence a universal characteristic class for  $G$ -principal  $\infty$ -bundles, represented by a smooth cocycle

$$\nabla_{\text{CS}}^0 : \mathbf{B}G \longrightarrow \mathbf{B}^{n+1}A .$$

Along the above lines we may think of the corresponding  $\mathbf{B}^n A$ -principal  $\infty$ -bundle over  $\mathbf{B}G$  as a *universal  $\infty$ -Chern-Simons bundle*. By example 2.1.31 this is the delooped  $\infty$ -group extension which is classified by  $\nabla_{\text{CS}}^0$  regarded as an  $\infty$ -group cocycle. The looping of this cocycle exists

$$\nabla_{\text{WZW}}^0 := \Omega \nabla_{\text{CS}}^0 : G \longrightarrow \mathbf{B}^n A .$$

and modulates a  $\mathbf{B}^{n-1}A$ -principal bundle over the  $\infty$ -group  $G$  itself: the  $\infty$ -group extension itself that is classified by  $\nabla_{\text{CS}}^0$  according to example 2.1.31. This is the corresponding WZW  $\infty$ -bundle.

For example, for the case that  $G \in \text{Grp}(\text{Smooth}\infty\text{Grpd})$  is a compact Lie group and  $A = U(1)$  is the smooth circle group, then by example 2.1.21 there is an essentially unique refinement of every integral cohomology class  $k \in H^4(BG, \mathbb{Z})$  to such a smooth cocycle  $\nabla_{\text{CS}}^0 : \mathbf{B}G \rightarrow \mathbf{B}^3U(1)$ . This  $k$  is the *level* of  $G$ -Chern-Simons theory and  $\nabla_{\text{CS}}^0$  modulates the corresponding higher prequantum bundle of 3d  $G$ -Chern-Simons theory as in 2.6.1 above. Moreover, the looping  $\nabla_{\text{WZW}}^0 \simeq \Omega \nabla_{\text{CS}}^0$  modulates the “WZW gerbe”, as discussed there.

Now restrict attention to the next higher example of such pairs of higher Chern-Simons/higher WZW bundles, as seen by the tower of examples induced by the smooth Whitehead tower of  $\mathbf{B}O$  in example 2.1.17: the universal Chern-Simons 7-bundle on the smooth String-2 group and the corresponding Wess-Zumino-Witten 6-bundle on String itself.



To motivate this as part of a theory of physics, first consider a simpler example of a 7-dimensional Chern-Simons type theory, namely the cup-product  $U(1)$ -Chern-Simons theory in 7 dimensions, for which the “holographic” relation to an interesting 6d theory is fairly well understood. This is the theory whose de-transgression is given [13] by the higher prequantum 7-bundle on the universal moduli 3-stack  $\mathbf{B}^3U(1)_{\text{conn}}$  of  $\mathbf{B}^2U(1)$ -principal connections that is modulated by the smooth and differential refinement of the cup product  $\cup$  in ordinary differential cohomology:

$$\begin{array}{ccc}
\mathbf{B}^3U(1)_{\text{conn}} & \xrightarrow{(-)\dot{\cup}(-)} & \mathbf{B}^7U(1)_{\text{conn}} \\
\downarrow u_{\mathbf{B}^3U(1)} & & \downarrow u_{\mathbf{B}^7U(1)} \\
\mathbf{B}^3U(1) & \xrightarrow{(-)\cup(-)} & \mathbf{B}^7U(1) \\
\downarrow f & & \downarrow f \\
K(\mathbb{Z}_4) & \xrightarrow{(-)\cup(-)} & K(\mathbb{Z}_8)
\end{array}
\quad \left| \begin{array}{l} \nabla_{7\text{AbCS}} \\ \nabla_{7\text{AbCS}}^0 \\ \int \nabla_{7\text{AbCS}}^0 \end{array} \right. .$$

(Or rather, the theory to consider for the full holographic relation is a quadratic refinement of this cup pairing. The higher geometric refinement of this we discussed in [51], but in the present discussion we will suppress this, for simplicity).

While precise and reliable statements are getting scarce as one proceeds with the physics literature into the study of these systems, the following four seminal physics articles seem to represent the present understanding of the story by which this 7d theory is related to a 6d theory in higher generalization of how 3d Chern-Simons theory is related to the 2d WZW model.

1. In [52] it was argued that the space of states that the (ordinary) geometric quantization of  $\nabla_{7\text{AbCS}}$  assigns to a closed 6d manifold  $\Sigma$  is naturally identified with the space of conformal blocks of a self-dual 2-form higher gauge theory on  $\Sigma$ . Moreover, this 6d theory is part of the worldvolume theory of a single M5-brane and the above 7d Chern-Simons theory is the abelian Chern-Simons sector of the 11-dimensional supergravity Lagrangian compactified to a 7-manifold whose boundary is the 6d M5-brane worldvolume.
2. Then in [53] a more general relation between the 6d theory and 11-dimensional supergravity compactified on a 4-sphere to an asymptotically anti-de Sitter space was argued for. This is what is today called  $\text{AdS}_7/\text{CFT}_6$ -duality, a sibling of the  $\text{AdS}_5/\text{CFT}_4$ -duality which has received a large amount of attention since then.
3. As a kind of synthesis of the previous two items, in [54] it is argued for both  $\text{AdS}_5/\text{CFT}_4$  and  $\text{AdS}_7/\text{CFT}_6$  the conformal blocks on the CFT-side are obtained already by keeping on the supergravity side *only* the Chern-Simons terms inside the full supergravity action.
4. At the same time it is known that the abelian Chern-Simons term in the 11-dimensional supergravity action relevant for  $\text{AdS}_7/\text{CFT}_6$  is not in general just the abelian Chern-Simons term  $\nabla_{7\text{AbCS}}$  considered in the above references: more accurately it receives Green-Schwarz-type quantum corrections that make it a *nonabelian* Chern-Simons term [55].

In [33] we observed that these items together, taken at face value, imply that more generally it must be the quantum-corrected nonabelian 7d Chern-Simons Lagrangian inside 11-dimensional supergravity which is relevant for the holographic description of the 2-form sector of the 6d worldvolume theory of M5-branes. (See [19] for comments on this 6d theory as an extended QFT related to extended 7d Chern-Simons theory.) Moreover, in [51] we observed that the natural lift of the “flux quantization condition” [52] – which is an *equation* between cohomology classes of fields in 11d-supergravity – to moduli stacks of fields (hence to higher prequantum geometry) is given by the corresponding *homotopy pullback* of these moduli fields, as usual in homotopy theory. We showed that this homotopy pullback is the smooth moduli 2-stack  $\mathbf{B}\text{String}_{\text{conn}}^{2\mathbf{a}}$  of



twisted String-principal 2-connections, unifying the Spin-connection (the field of gravity) and the 3-form  $C$ -field into a single higher gauge field in higher prequantum geometry.

The nonabelian 7-dimensional Chern-Simons-type Lagrangian on String-2-connections obtained this way in [33] is the sum of some cup product terms and one indecomposable term. Moreover, the refinement specifically of the indecomposable term to higher prequantum geometry is the stacky and differential refinement  $\frac{1}{6}\hat{\mathbf{p}}_2$  of the universal fractional second Pontryagin class  $\frac{1}{2}p_2$ , which was constructed in [8] as reviewed in 2.6.2 above:

$$\begin{array}{ccc}
 \mathbf{B}\text{String}_{\text{conn}} & \xrightarrow{\frac{1}{6}\hat{\mathbf{p}}_2} & \mathbf{B}^7U(1)_{\text{conn}} \\
 \downarrow u_{\mathbf{B}\text{String}} & & \downarrow p_{\mathbf{B}^7U(1)} \\
 \mathbf{B}\text{String} & \xrightarrow{\frac{1}{6}\mathbf{p}_2} & \mathbf{B}^7U(1) \\
 \downarrow f & & \downarrow f \\
 BO\langle 8 \rangle & \xrightarrow{\frac{1}{6}p_2} & K(\mathbb{Z}, 8)
 \end{array}
 \quad \left| \begin{array}{l}
 \nabla_{7\text{CS}} \\
 \nabla_{7\text{CS}}^0 \\
 \int \nabla_{7\text{CS}}^0
 \end{array} \right.
 .$$

Quite independently of whatever role this extended 7d Chern-Simons theory has as a sector in  $\text{AdS}_7/\text{CFT}_6$  duality, this is the natural next example in higher prequantum theory after that of 3d Spin-Chern-Simons theory.

In [8] it was shown that the prequantum 7-bundle of this nonabelian 7d Chern-Simons theory over the moduli stack of its instanton sectors, hence over  $\mathbf{B}\text{String}$ , is the delooping of a smooth refinement of the *Fivebrane group* [32] to the smooth Fivebrane 6-group of example 2.1.17:

$$\begin{array}{ccc}
 \mathbf{B}\text{Fivebrane} & & \\
 \downarrow & & \\
 \mathbf{B}\text{String} & \xrightarrow{\nabla_{7\text{CS}}^0} & \mathbf{B}^7U(1) .
 \end{array}$$

Moreover, by the above general discussion this induces a WZW-type 6-bundle over the smooth String 2-group itself, whose total space is the Fivebrane group itself

$$\begin{array}{ccc}
 \text{Fivebrane} & & \\
 \downarrow & & \\
 \text{String} & \xrightarrow{\nabla_{6\text{WZW}}^0} & \mathbf{B}^6U(1)
 \end{array}
 .$$

Therefore, in view of the discussion in 2.6.1, it is natural to expect a 6-dimensional higher analog of traditional 2d WZW theory whose underlying higher prequantum 6-bundle is  $\nabla_{6\text{WZW}}$ . However, the lift of this discussion from just instanton sectors to the full moduli stack of fields is more subtle than in the 3d/2d case and deserves a separate discussion elsewhere. (This is ongoing joint work with Hisham Sati.)

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This article here refers to details that appear separately in the following writeups:

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